Sensitivity lower bounds from linear dependencies

Sensitivity, Query Complexity, Communication Complexity and Fourier Analysis of Boolean Function, ISI Kolkata

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Timeline

1992 Block sensitivity is at most quadratic in the degree (Nisan-Szegedy).

1992 Showing degree is at most some polynomial in sensitivity is equivalent to a lower bound on the maximum degree of any large enough induced subgraph (Gotsman-Linial).

1994 Conjecture: block sensitivity and sensitivity are polynomially related (Nisan-Szegedy).

2019 The maximum degree of any large enough induced subgraph of a hypercube is at least $\sqrt{n}$ (Huang).

Our work improves upper bound on the degree by connecting it to 1-sensitivity and 0-sensitivity of the function.

Previous results : $\mathrm{bs}(f) \leq \deg(f)^2$, $\deg(f) \leq (\mathrm{s}(f))^2$

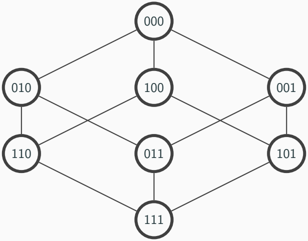
Our result: $\deg(f) \leq \mathrm{s}_0(f) \cdot \mathrm{s}_1(f)$

Boolean Hypercube

- Boolean hypercube H_n

  $V(H_n) = \{0,1\}^n$

  $u \sim v$ iff they differ exactly at one bit



- U^{odd} - vertices of H_n with odd Hamming weight
- U^{even} - vertices of H_n with even Hamming weight
- Boolean function - $f : \{0,1\}^n \mapsto \{0,1\}$

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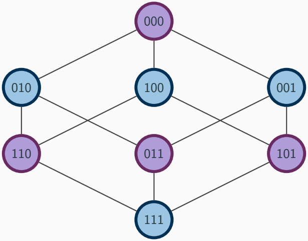

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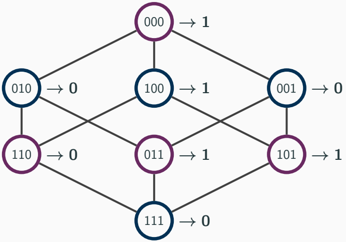

- U^{odd} - vertices of H_n with odd Hamming weight
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Sensitivity of a function

- *Sensitivity* of an input x w.r.t f:

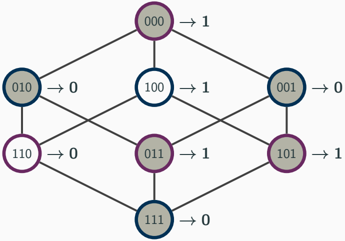
$$\mathrm{s}(f, x) = |\{i \mid f(x) \neq f(x^{(i)})\}|$$

- *Sensitivity* of a function:

$$\mathrm{s}(f) = \max_{x \in \{0,1\}^n} \mathrm{s}(f, x)$$

- *z-sensitivity* of a function:

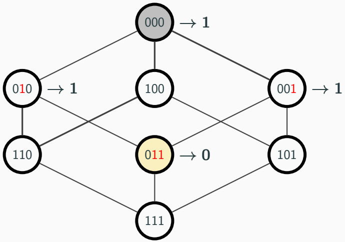
$$\mathrm{s}_z(f) = \max_{x \in f^{-1}(z)} \mathrm{s}(f, x)$$

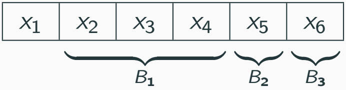


Block sensitivity of a function

- For a $B \subseteq [n]$, an input x is B-sensitive w.r.t f, if $f(x^B) \neq f(x)$.
- *Block sensitivity* of x w.r.t f, $\mathrm{bs}(f, x)$ is the maximum number of disjoint blocks $B_i \subseteq [n]$ such that $\forall i$, x is B_i-sensitive.
- Block sensitivity of f:

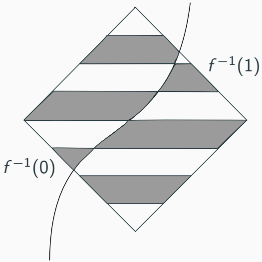
$$\mathrm{bs}(f) = \max_{x \in \{0,1\}^n} \mathrm{bs}(f, x)$$





Polynomial degree

- A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ *represents* a Boolean function f $\forall x \in \{0,1\}^n,\ p(x) = f(x)$.
- The degree of the polynomial representing the Boolean function f is its *degree*, $\deg(f)$.
- *Parity-balanced* functions
  - Layer i of H_n has vertices of Hamming weight i
  - Shaded area: $f^{-1}(1) \cap U^{\text{even}}$ and $f^{-1}(0) \cap U^{\text{odd}}$
  - *Parity-balanced* iff shaded and unshaded area are equal



- **Theorem:** *(Shi-Yao)* f has full degree ($= n$) iff it is not parity-balanced.

Connections between various complexity measures

Theorem: *(Nisan-Szegedy, Tal)* For any Boolean function f, we have:

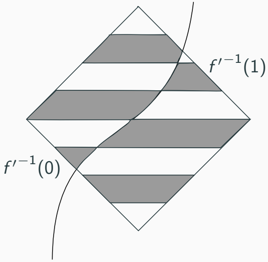
$$\mathrm{bs}(f) \leq \deg(f)^2$$

Theorem: *(Special case of Gotsman-Linial)*
The following are equivalent:

1. For any Boolean function f of degree d, $\mathrm{s}(f) \geq \sqrt{d}$.
2. For any induced subgraph G of H_n such that $|V(G)| \neq 2^{n-1}$, there exists a vertex of degree $\geq \sqrt{n}$ in either G or $H_n - G$.

Proof sketch of Gotsman-Linial

- Construct modified function f':
  - Take a monomial of degree d in polynomial representing f.
  - Set variables not in monomial to 0.
- Sensitivity $\mathrm{s}(f') \leq \mathrm{s}(f)$, but degree $\deg(f') = deg(f) = d$.
- For a function f' with full degree, f' is not parity-balanced.
- Degree of a vertex in subgraph induced on shaded vertices equals its sensitivity.



Huang's Proof of the Sensitivity Conjecture

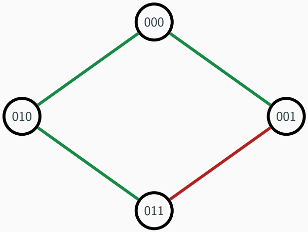
- Any induced subgraph on more than half the vertices in H_n has a vertex of degree at least $\sqrt{n}$.
- Proof is based on:
    - Specific choice of signed adjacency matrix of H_n with exactly two eigenvalues ($+\sqrt{n}$ and $-\sqrt{n}$).
    - Uses interlace theorem to determine the largest eigenvalue of any large enough principal submatrix of the signed adjacency matrix.
    - Maximum degree of a graph $\geq$ maximum eigenvalue.

Inductive Construction of Signed Adjacency Matrix

- σ: assignment of $+$ or $-$ signs to the edges of H_n.

$$\begin{aligned}\sigma(x,y) &= \pm 1 \quad \textit{if } (x,y) \in E \\ &= 0 \quad \textit{otherwise}\end{aligned}$$

- σ assigns every C_4 a negative sign.



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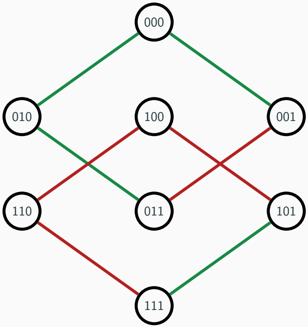

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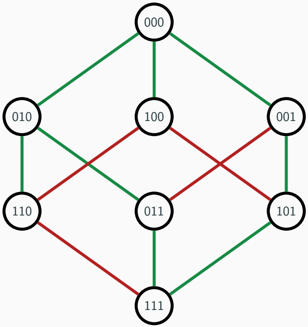
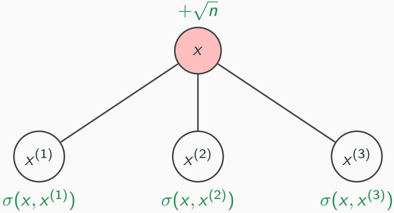


Alternate proof of Huang's result

- For every vertex x in the hypercube H_n, assign vectors x^+ and x^- labelled by $V(H_n)$.

$$x^{+/-} = \begin{pmatrix} \sigma(x, x^{(1)}) \\ \vdots \\ +/-\sqrt{n} \\ \sigma(x, x^{(i)}) \\ \vdots \\ 0 \\ \vdots \\ \sigma(x, x^{(n)}) \end{pmatrix} \begin{matrix} \\ \\ \longleftarrow x \\ \\ \\ \\ \\ \\ \end{matrix}$$

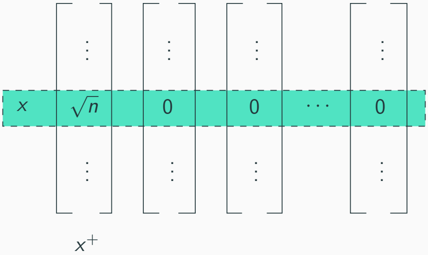


- We denote the subspaces generated by the vectors x^+ and x^- by V^+ and V^- resp.

Dimension of V^+ and V^-

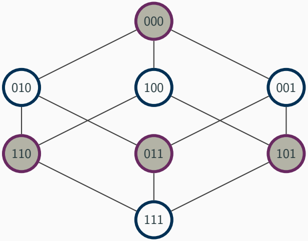
- Any set I^+ of vectors x^+ corresponding to an independent set in the hypercube is linearly independent.

Reason: x^+ is non-zero only at itself and its neighbours, but neighbours are not in the independent set.



- Since U^{odd} and U^{even} are independent sets,

$$\dim(V^+) \geq 2^{n-1}$$
$$\dim(V^-) \geq 2^{n-1}$$

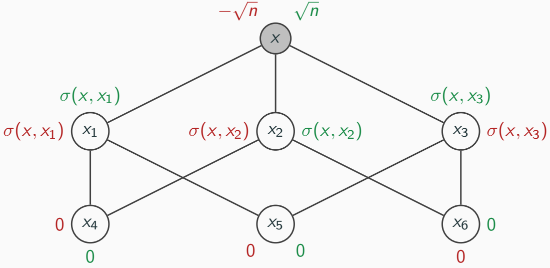


- Claim : $\dim(V^+) = \dim(V^-) = 2^{n-1}$.

Orthogonality of V^+ and V^-

- Case 1. $\underline{x = y}$

$$\langle x^+, x^- \rangle = -\sqrt{n} \cdot \sqrt{n} + \sum_{z : z \sim x} \big(\sigma(x, z)\big)^2 = -n + n = 0$$

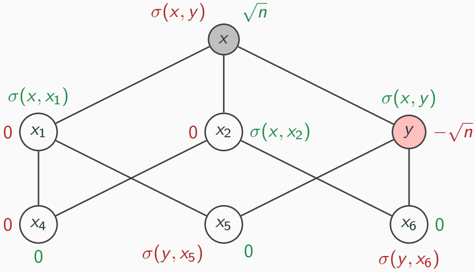


Case 1

Orthogonality of V^+ and V^-

- Case 2. $\underline{x \sim y}$,

$$\langle x^+, y^- \rangle = \sqrt{n}\sigma(x,y) - \sqrt{n}\sigma(x,y) = 0$$



Case 2

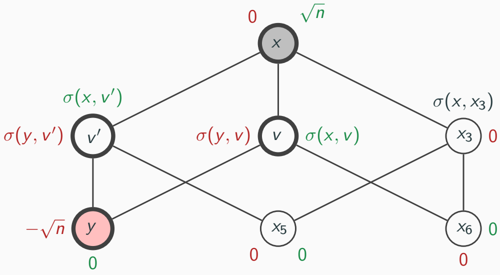
Orthogonality of V^+ and V^-

- Case 3. $d_H(x, y) = 2$:

  $\exists$ vertices v, v' such that x, v, y, v' form a C_4

  $$\langle x^+, y^- \rangle = \sigma(x, v)\sigma(v, y) + \sigma(x, v')\sigma(y, v')$$

  $\sigma(x, v)\sigma(v, y) + \sigma(x, v')\sigma(y, v') = 0$ as σ assigns negative sign to every C_4.



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  $\sigma(x, v)\sigma(v, y) + \sigma(x, v')\sigma(y, v') = 0$ as σ assigns negative sign to every C_4.
- Case 4. $\underline{d_H(x, y) \geq 3}$, $\langle x^+, y^- \rangle$ is trivially 0.
- Since $V^+ \perp V^-$ and $\dim(V^+) + \dim(V^-) \leq 2^n$,

  $$\dim(V^+) = \dim(V^-) = 2^{n-1}$$

Linear dependencies: An observation and Huang's bound

The observation: If the vectors $\{x_1^+, x_2^+, \ldots, x_k^+\}$ have a linear dependency, the subgraph induced on the corresponding vertices $\{x_1, x_2, \ldots, x_k\}$ of H_n has a vertex of degree at least $\sqrt{n}$.

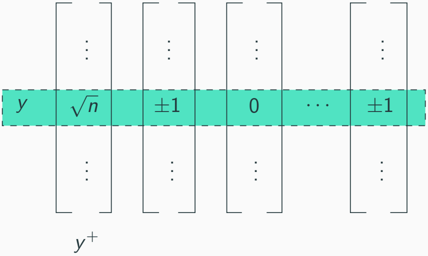
Proof:

Let

$$\sum a_x x^+ = 0$$

with $a_x \neq 0$ and let

$$|a_y| = \max_x |a_x|$$



At coordinate y,

$$|a_y|\sqrt{n} \leq \sum_{x \sim y} |\sigma(y,x)a_x| \leq \sum_{x \sim y} |a_y|$$

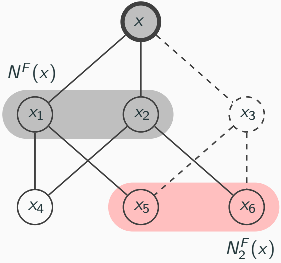
Thus y has at least $\sqrt{n}$ neighbours. □

More structural information from linear dependencies

- F is a non-trivial linear dependency if

$$\sum_{u \in U^{\mathbf{odd}}} a_u u^+ = \sum_{v \in U^{\mathbf{even}}} b_v v^+$$

  where not all of the a_u and b_v are 0.
- H_F: subgraph induced by the vertices of H_n which have a nonzero coefficient in F.
- $N^F(x)$: neighbours of x in H_F.
- Degree of x in H_F, $d_F(x) = |N^F(x)|$.
- $N_2^F(x)$: vertices $y \in H_F$ at distance 2 from x such there is a unique 2-path in H_F from each of them to x.



$\boldsymbol{F : 2\sqrt{3}x_1^+ + \sqrt{3}x_2^+ = 3x^+ + x_4^+ + 2x_5^+ - x_6^+}$

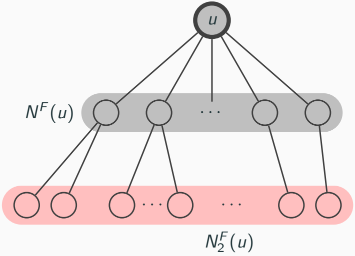
Improving Huang's bound

Theorem 1:
Given a nontrivial linear dependency relation F, on a subset of vertices in H_n there exist vertices $u \in U^{\text{odd}}$ and $v \in U^{\text{even}}$ in H_F such that

$$|N^F(u)| + |N_2^F(u)| \geq n$$
$$|N^F(v)| + |N_2^F(v)| \geq n.$$

Corollary 1: If F is a nontrivial linear dependency, there exists an edge (u, v) in H_F such that

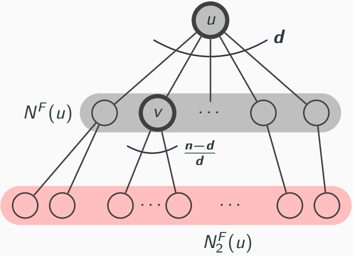
$$d_F(u) \times d_F(v) \geq n$$



Proof of Corollary 1

Corollary 1: If F is a nontrivial linear dependency, there exists an edge (u, v) in H_F such that $d_F(u) \times d_F(v) \geq n$

- Let $d_F(u) = d$, $\exists\ v \sim u$ such that v has $\geq \frac{n-d}{d}$ neighbours in $N_2^F(u)$.
- Thus $d_F(v) \geq \frac{n-d}{d} + 1 = \frac{n}{d}$



Corollary 2: For any Boolean function f, $\deg(f) \leq s_0(f)s_1(f)$.
Proof: By definition of the induced subgraph, $f(u) \neq f(v)$ for $u \sim v$. Thus $s_0(f)s_1(f) \geq d = \deg(f)$. □

Lemma

The key lemma in the proof of Theorem 1

Lemma: Given a linear dependency $F : \sum_{u \in U^{\mathbf{odd}}} a_u u^+ = \sum_{v \in U^{\mathbf{even}}} b_v v^+$,

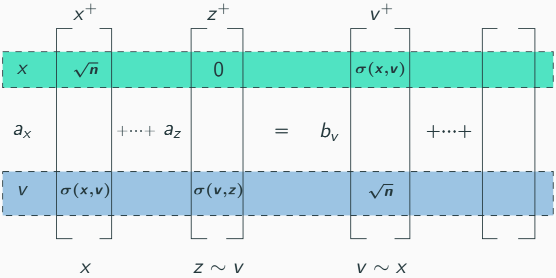
$\forall x \in H_F \cap U^{\text{odd}}$ and $\forall y \in H_F \cap U^{\text{even}}$,

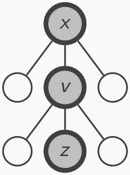
$$(n - d_F(x))a_x = \sum_{z \in N_{\mathbf{2}}^F(x)} \hat{\sigma}_F(x, z)a_z,$$

$$(n - d_F(y))b_y = \sum_{t \in N_{\mathbf{2}}^F(y)} \hat{\sigma}_F(t, y)b_t.$$

Here $\hat{\sigma}_F(x, y) = \sigma(x, z)\sigma(z, y)$ where z is the unique common neighbour of x and y in H_F.

Proof of Lemma





$$F : \sum_{u \in U^{\mathbf{odd}}} a_u u^+ = \sum_{v \in U^{\mathbf{even}}} b_v v^+$$

In the row for any $x \in U^{\text{odd}}$:

$$\sqrt{n} a_x = \sum_{v \in N^F(x)} \sigma(x, v) b_v$$

For any $v \in U^{\text{even}}$:

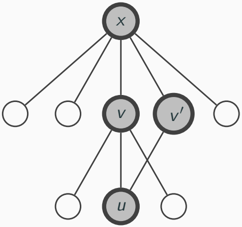
$$\sum_{u \in N^F(v)} \sigma(u, v) a_u = \sqrt{n} b_v$$

Proof of Lemma

$$na_x = \sum_{v \in N^F(x)} \sum_{u \in N^F(v)} \sigma(x,v)\sigma(v,u)a_u$$

On examining the R.H.S. we see:

- Case 1: $\underline{x = u}$, $\sigma(x,v)\sigma(v,u) = (\sigma(x,v))^2 = 1.$
- Case 2: $\underline{x \neq u}$,
  4-cycle: $(x, v, u, v') \implies \sigma(x,v)\sigma(v,u) + \sigma(x,v')\sigma(v',u) = 0$



$$na_x = d_F(x)a_x + \sum_{u \in N_2^F(x)} \hat{\sigma}_F(x,u)a_u$$

On rearranging,

$$(n - d_F(x))a_x = \sum_{u \in N_2^F(x)} \hat{\sigma}_F(x,u)a_u$$

Proof of Theorem 1

Theorem 1 claims the existence of two vertices one of odd and even parity having at least n neighbours in N^F and N_2^F.

$$|N^F(u)| + |N_2^F(u)| \geq n$$

Proof.

Let $|a_x| = \max_{z \in U^{\mathbf{odd}}} \{|a_z|\}$. Lemma gives

$$(n - d_F(x))a_x = \sum_{z \in N_{\mathbf{2}}^F(x)} \hat{\sigma}_F(x, z) a_z$$

Since $|a_z| \leq |a_x|$, there should be at least $n - d_F(x)$ values of a_z which are nonzero.

An analogous argument follows for the other part. $\square$

Application: Chakraborty function

- Chakraborty function $f_{n,k} : \{0,1\}^n \mapsto \{0,1\}$ evaluates x to 1 iff:

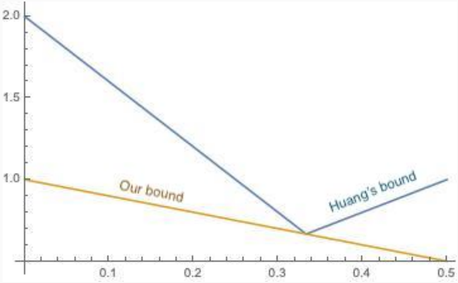
$$x \in \star \cdots \star 110^{k-2}(11111\star^{k-5})^{k-2}11111\,\star^{k-8}\,111\star \cdots \star$$

  for any $k \leq \sqrt{n}$.

- The sensitivity and block sensitivity are:

$$\mathrm{s}_0(f_{n,k}) = \frac{n}{k^2},\ \mathrm{s}_1(f_{n,k}) = k$$
$$\mathrm{bs}(f_{n,k}) = \frac{n}{k}$$

Degree of Chakraborty function



- Using Corollary 2, we give a new upper bound on the polynomial degree of $f_{n,k}$.

  For any $k \leq \sqrt{n}$:

$$\deg(f_{n,k}) \leq \frac{n}{k}$$

$$\deg(f_{n,k}) \geq \sqrt{\frac{n}{k}}$$ given by the Nisan-Szegedy Theorem.

Conclusion

- Spectral techniques give information about the immediate neighbourhood of vertices.
- We studied linear dependencies and their implications at distance 2.
- This improved Huang's degree bound to the product of 0-sensitivity and 1-sensitivity.
- Open question: improve the gap for Chakraborty's function.
- Remains to be studied : implications of linear dependencies at distances ≥ 3, similar techniques for bounds between other complexity measures.