Equality Cannot Simulate Randomness

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How Powerful is Randomness?

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Central Question in Theoretical CS.

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Central Question in Theoretical CS.

Theorem: $P \subseteq BPP \subseteq NP^{NP}$

Theorem: Under plausible assumptions $P = BPP$!!

Study Such Questions in Elementary Models.

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Decision Trees: $P = BPP$ (total functions)

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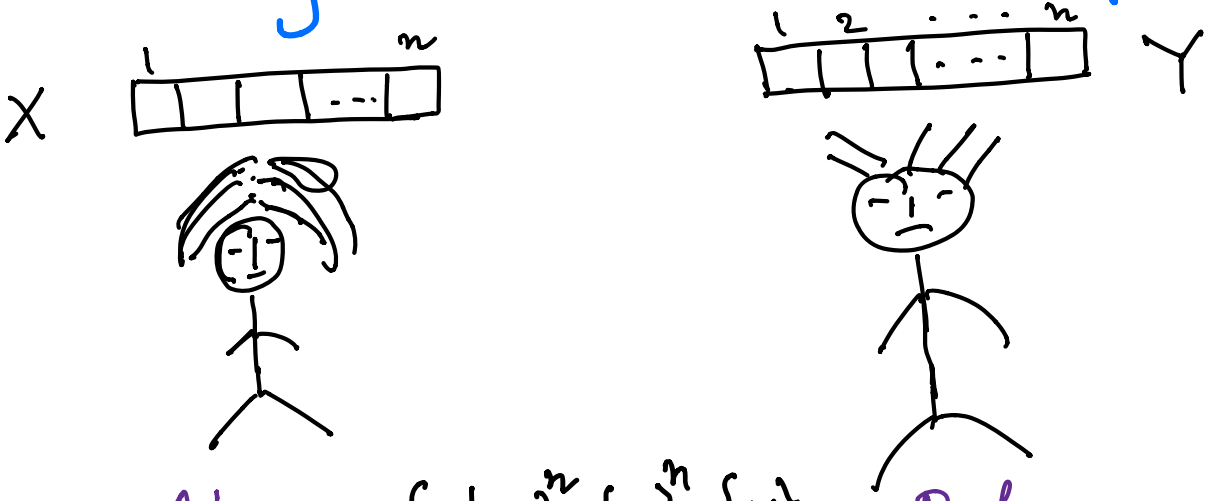
Decision Trees: $P = BPP$ (total functions)

Communication Complexity: $P \subsetneq BPP$.

2 - Party Common Model of Yao



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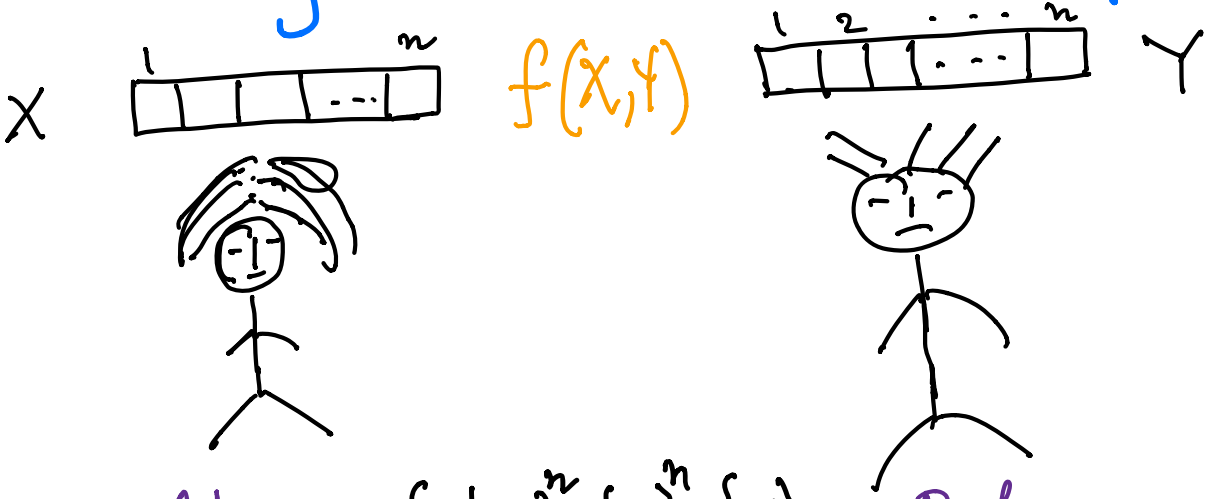

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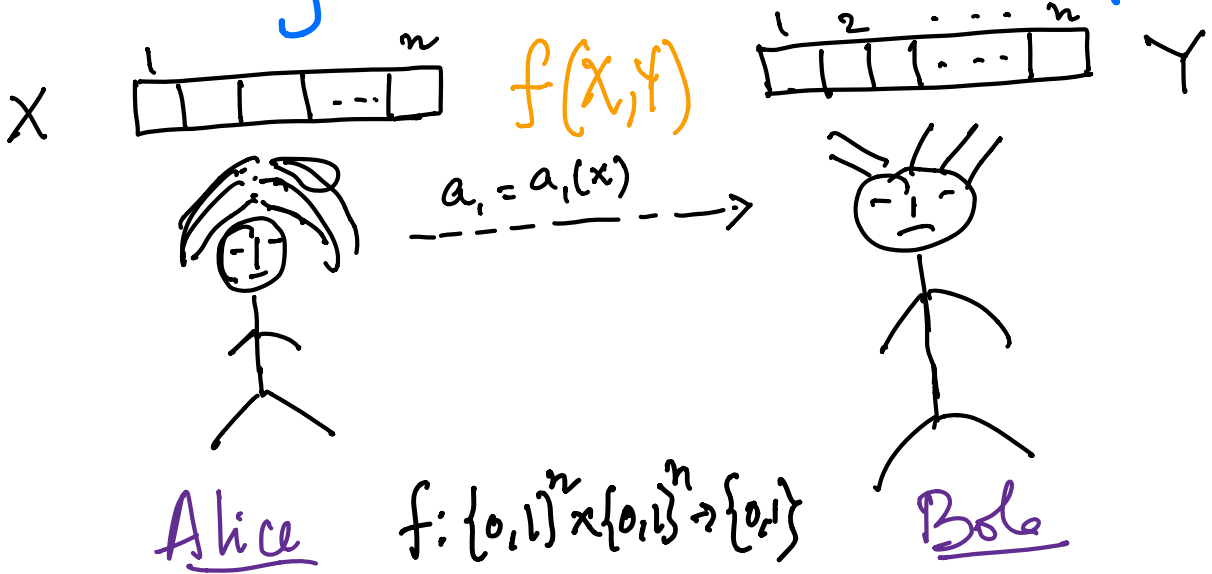

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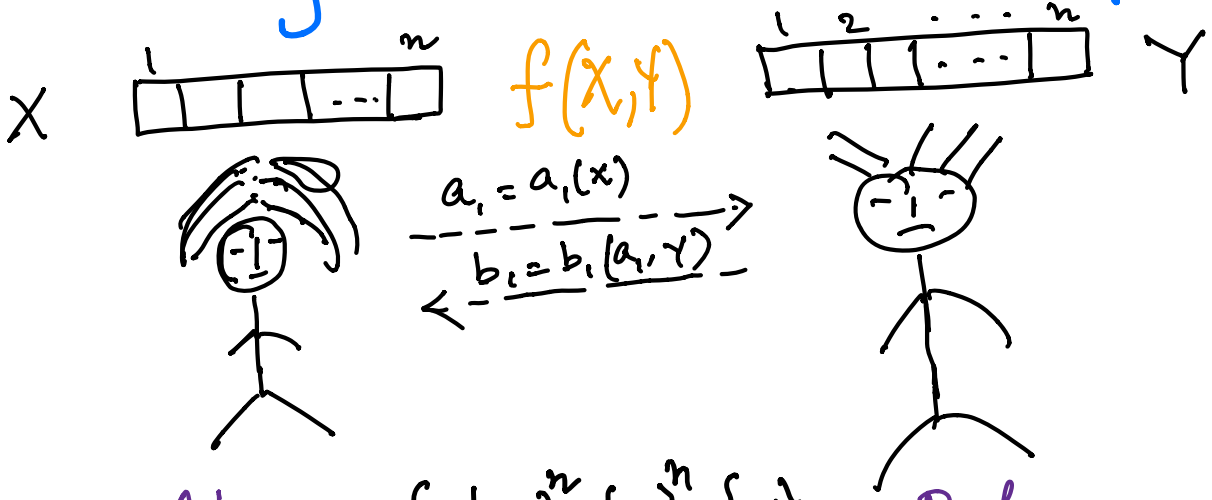

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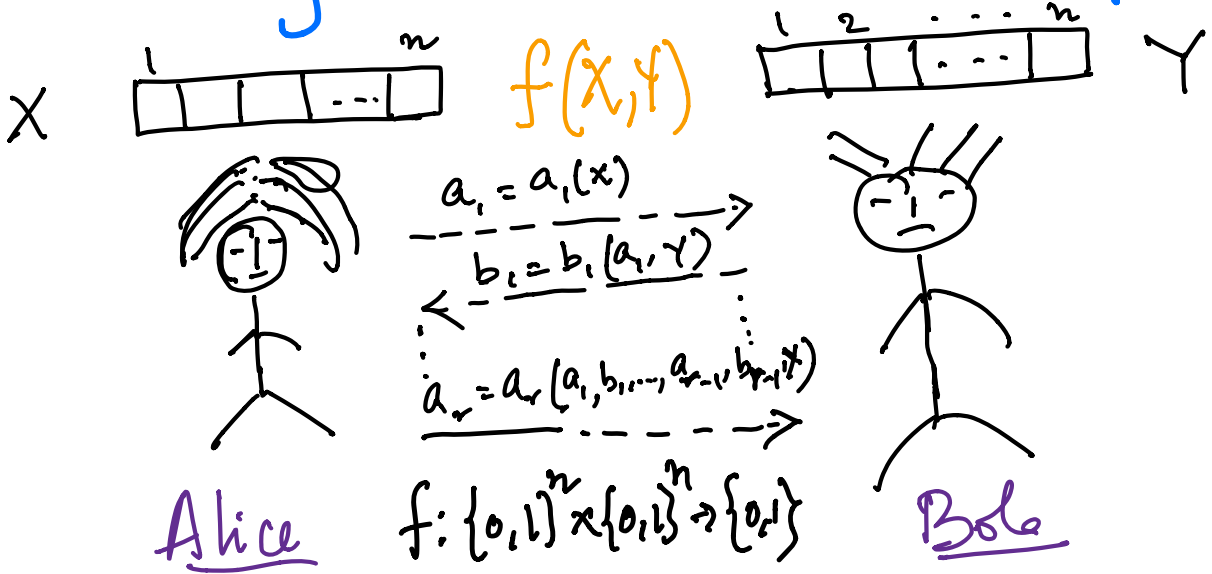
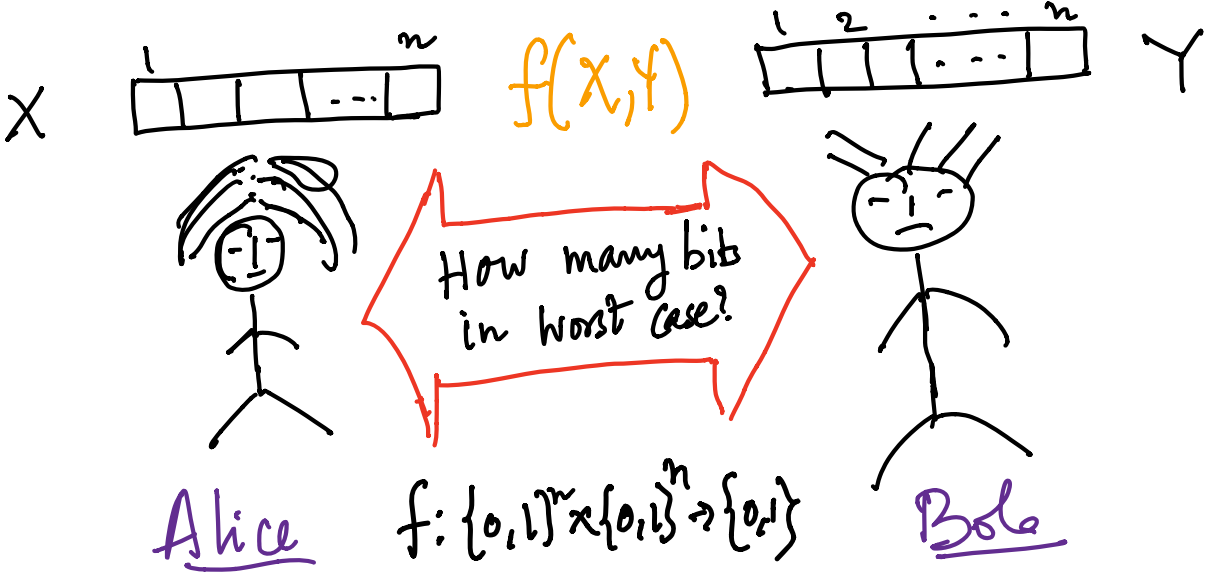


Deterministic Commn. Complexity



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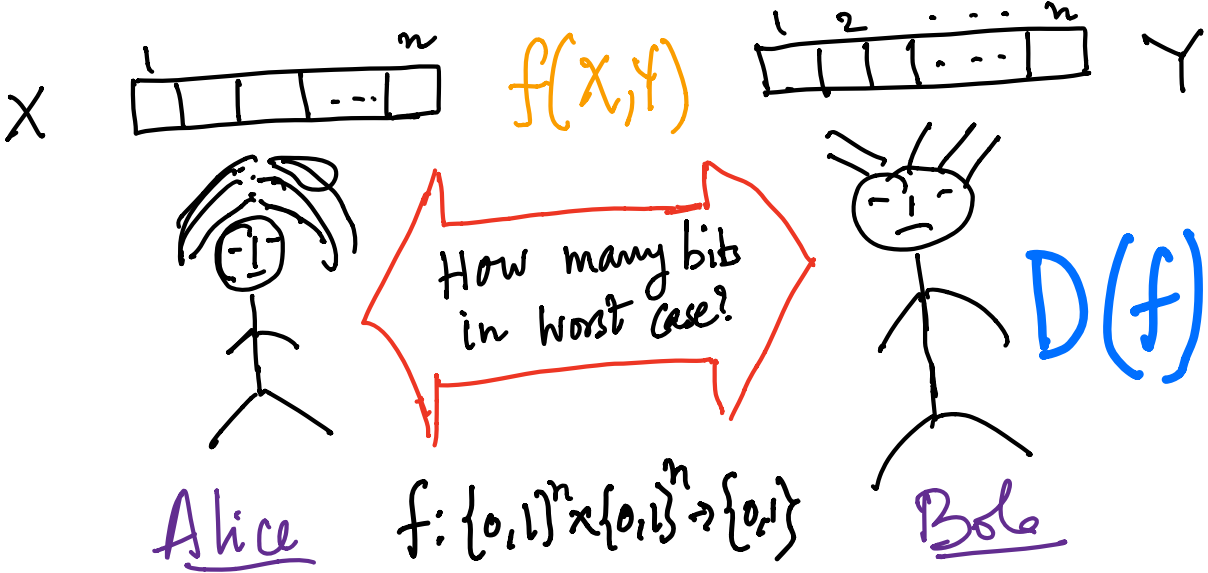

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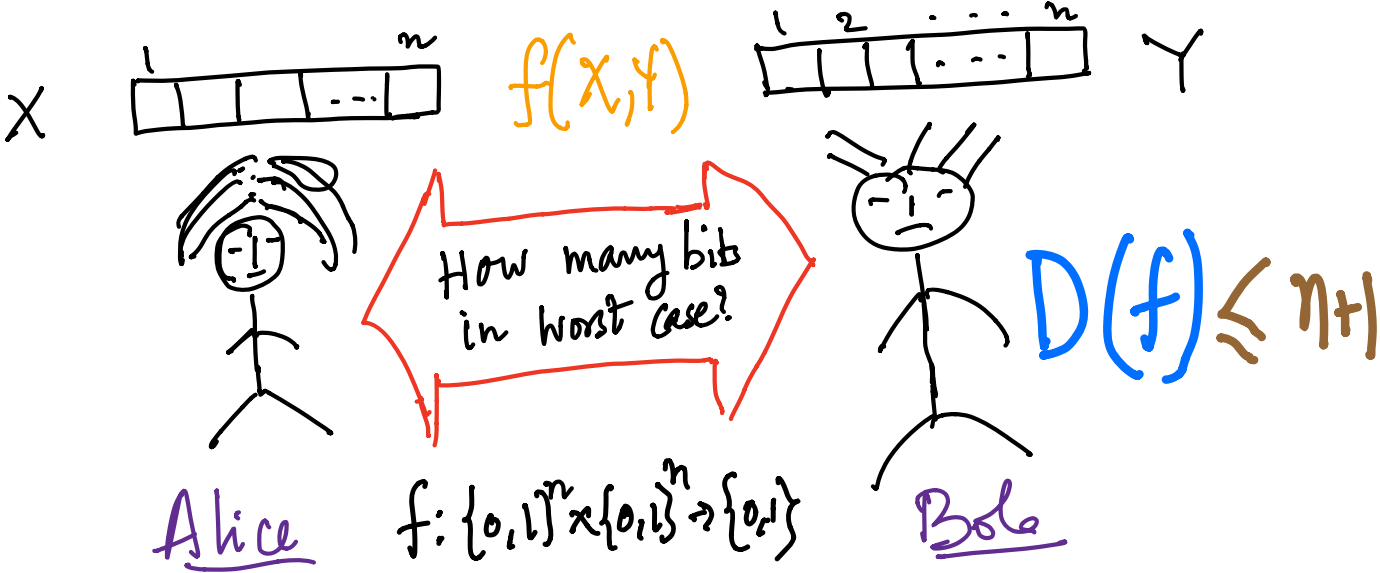
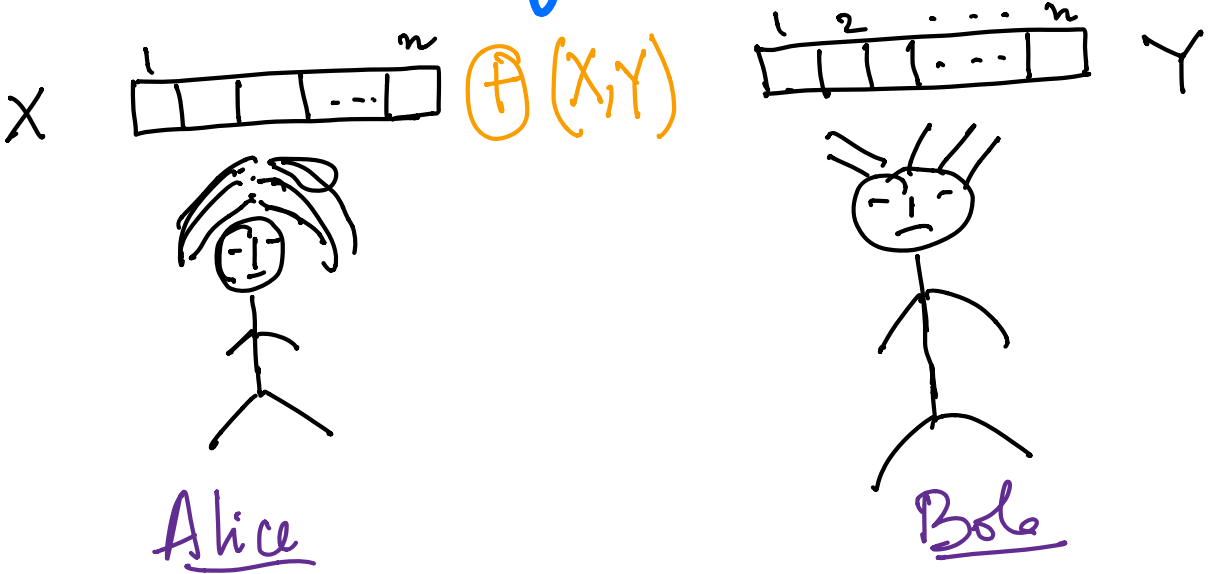


Easy Functions



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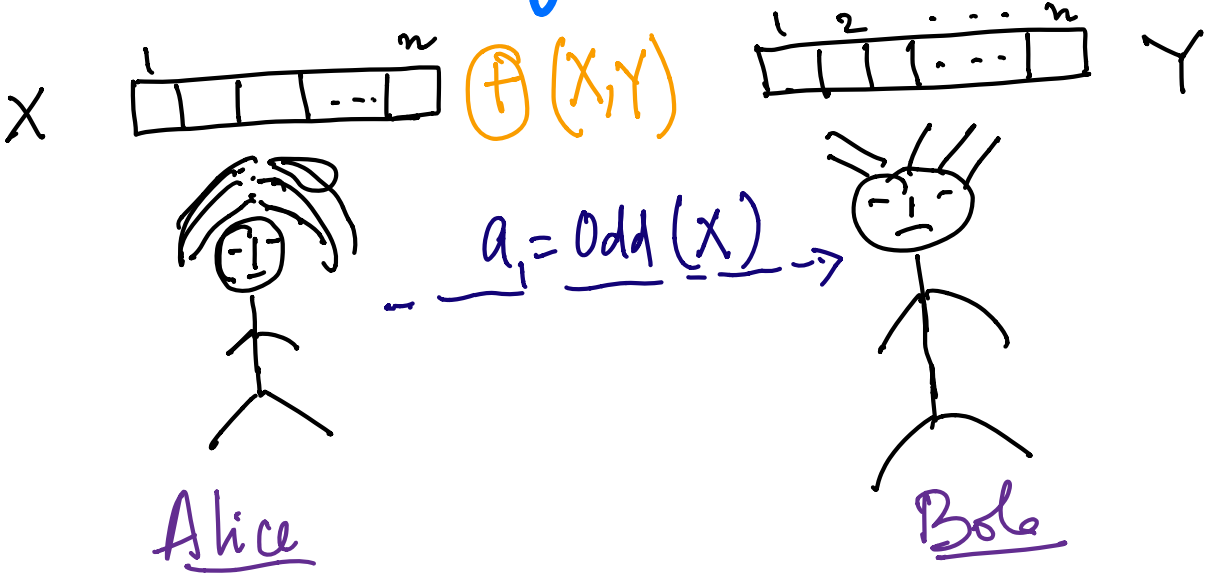

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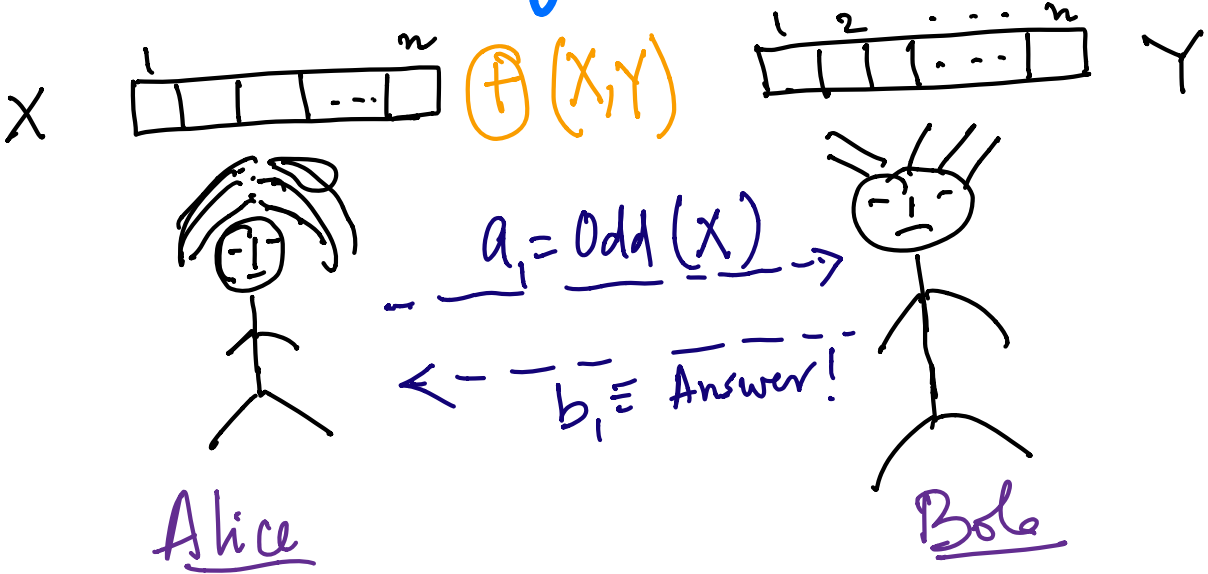

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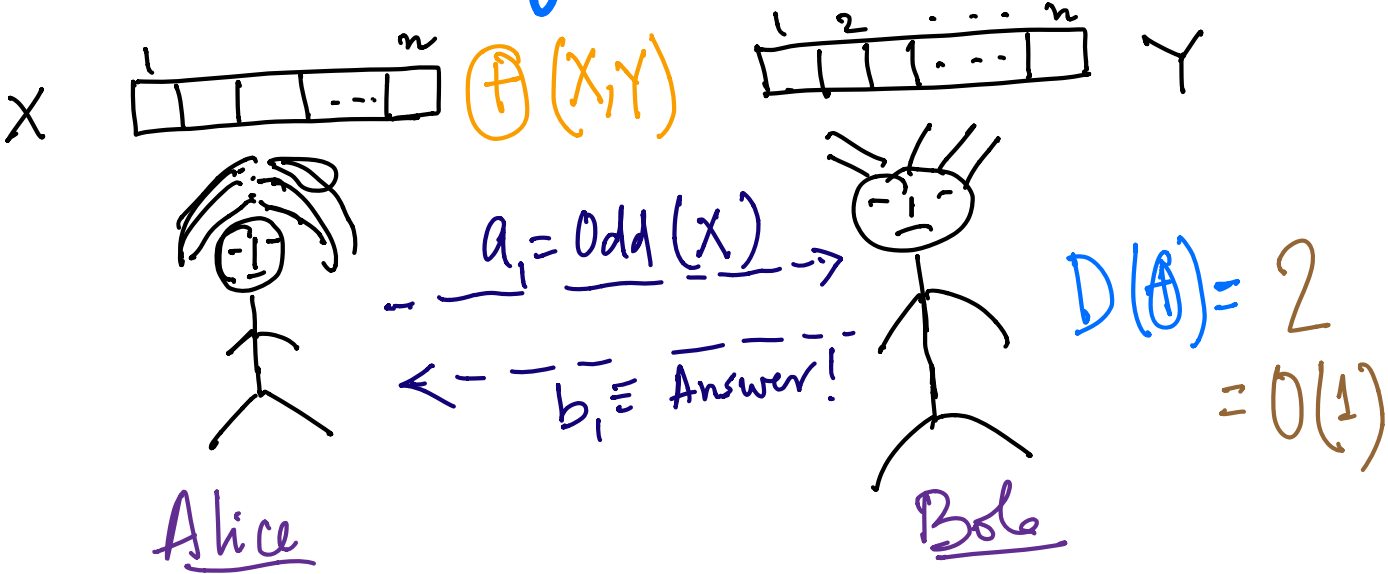
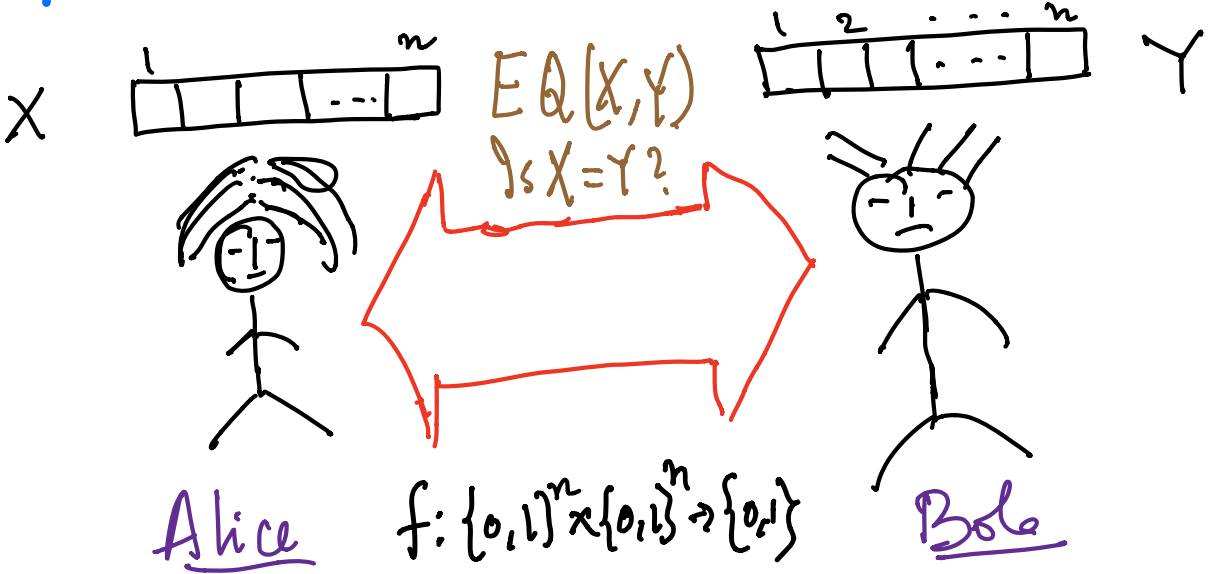


Hardness Of EQUALITY



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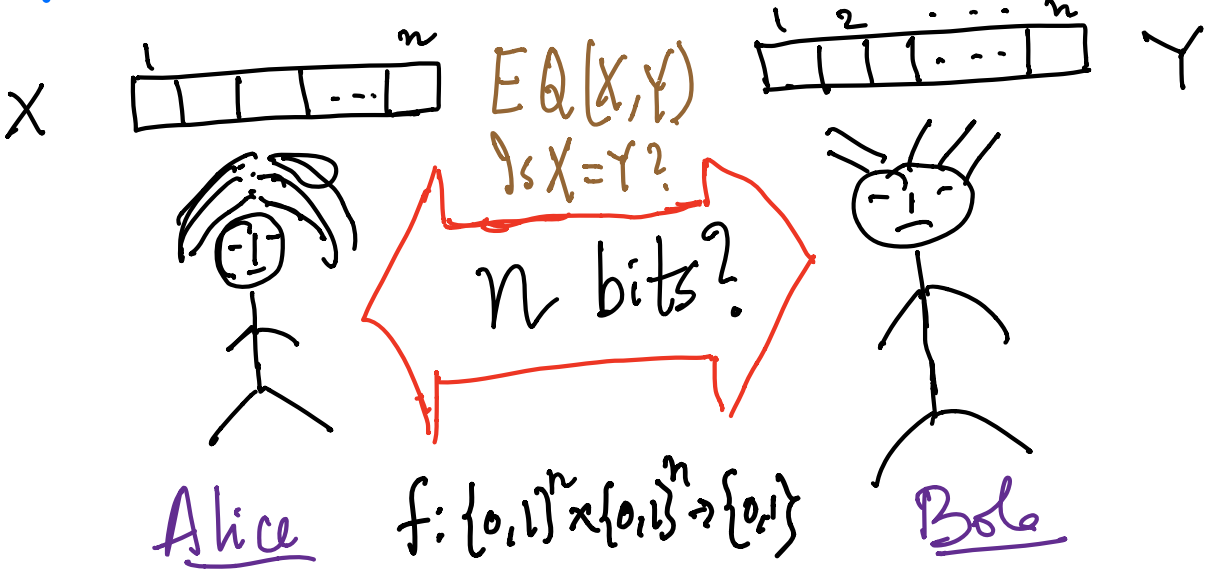

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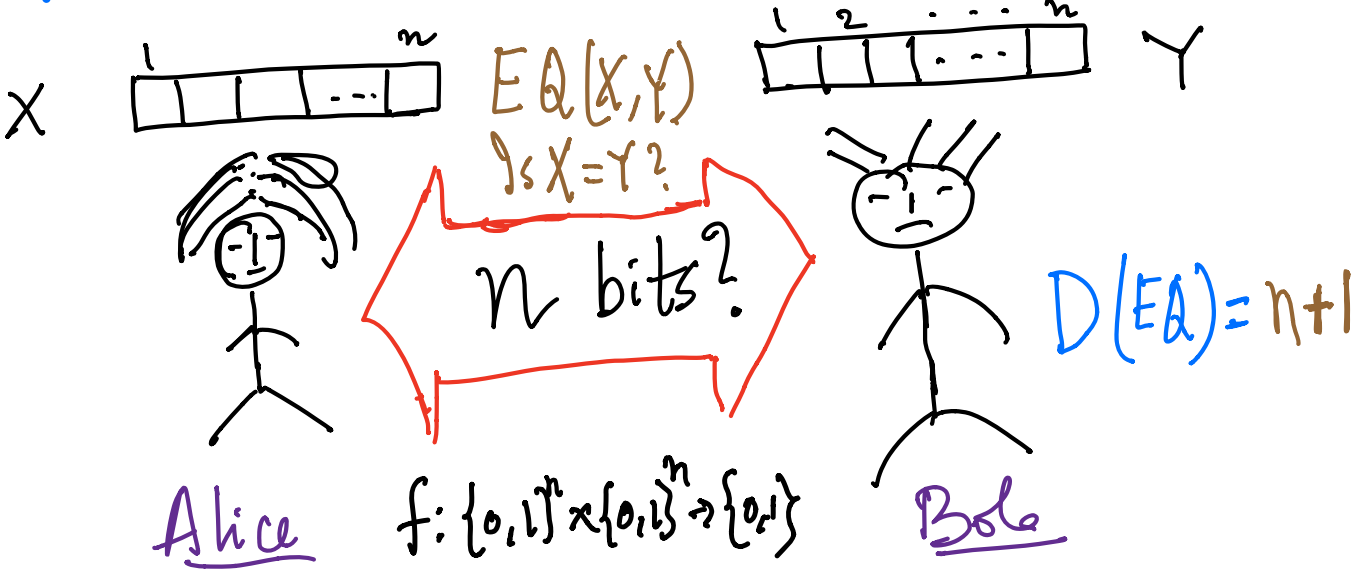

Easy Functions.

$$P \equiv \left\{ f_n \mid D(f_n) = (\log n)^{O(1)} \right\}$$

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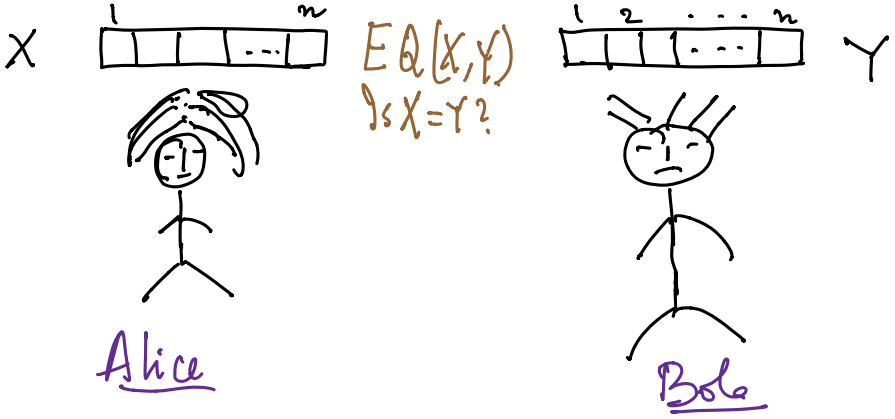
Examples:

PARITY, MAJORITY, SYMM....

Power Of Randomness

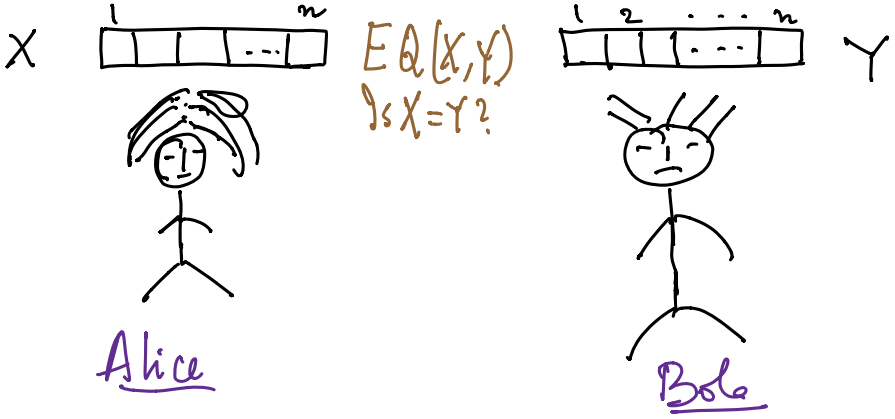
$D(EQ) = n+1$



Power Of Randomness

Randomness: 

$D(EQ) = n+1$



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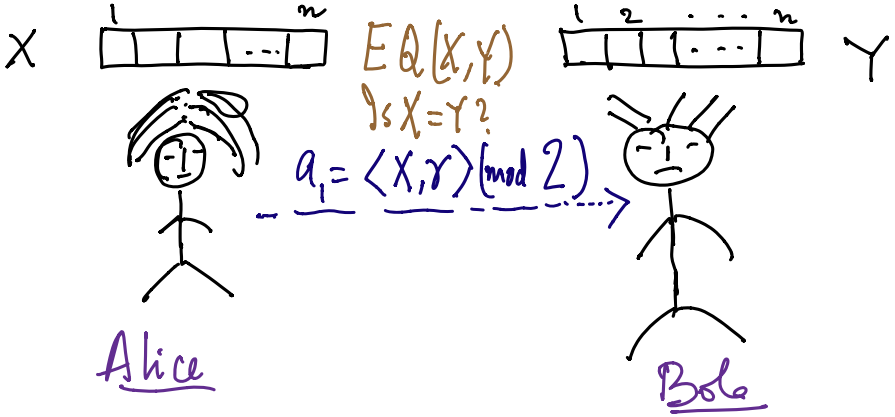

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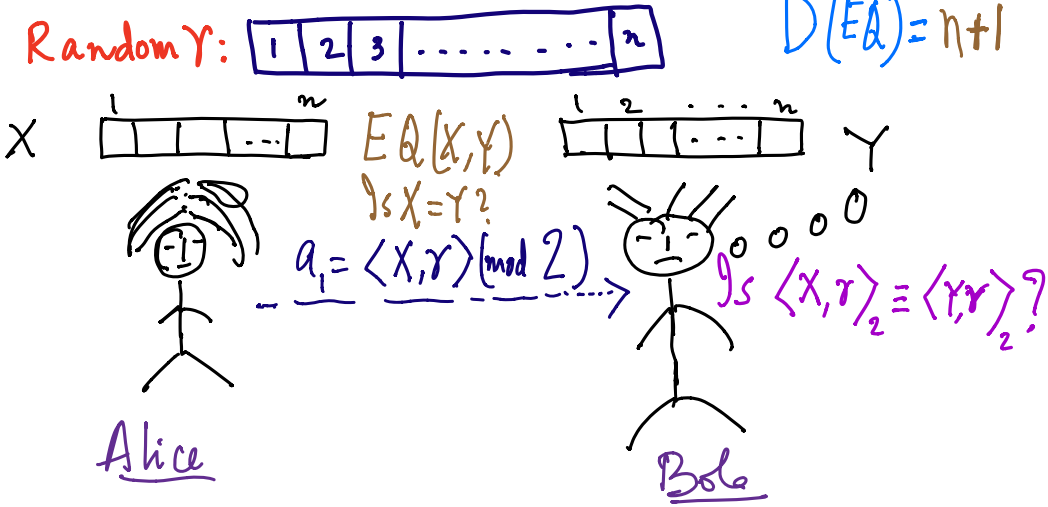

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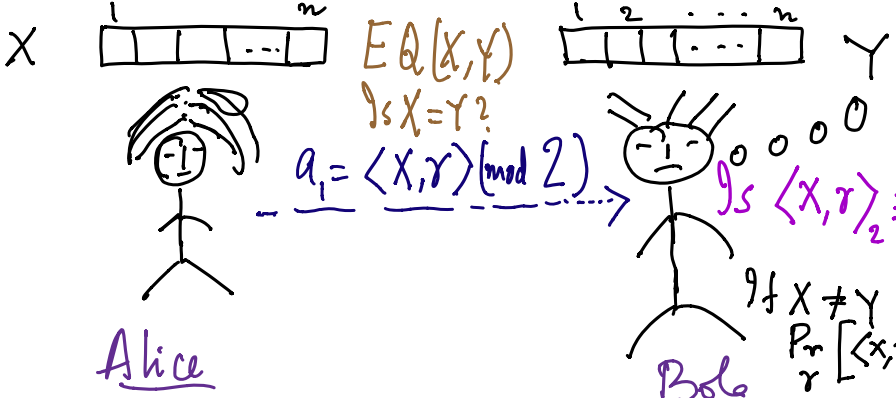

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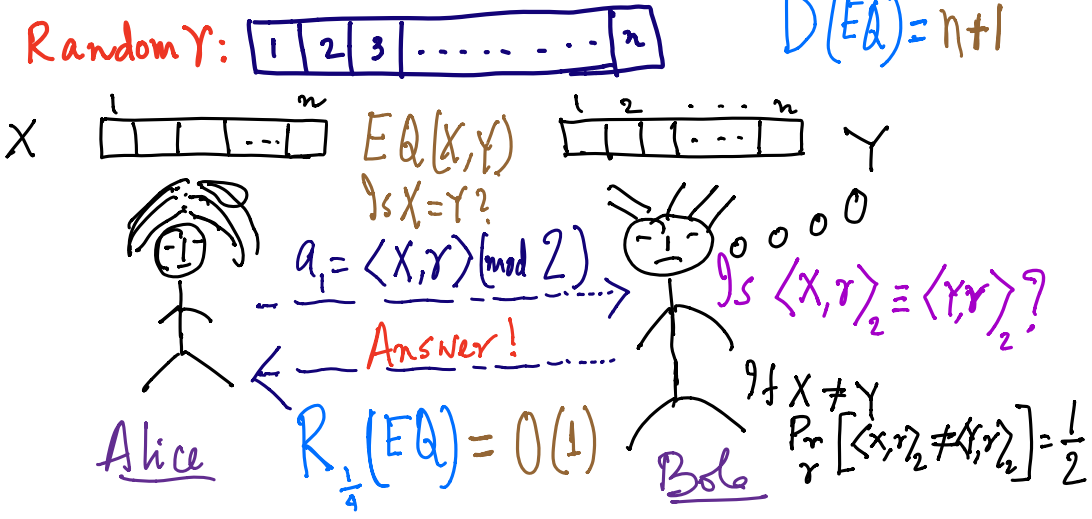

A Class Separation

$$BPP \equiv \left\{ f_n \mid R(f_n) = (\log n)^{O(1)} \right\}$$

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(via EQUALITY).

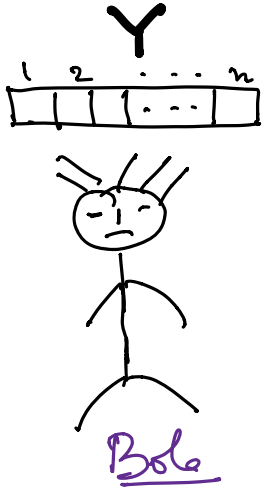
EQUALITY For Free

$$P^{EQ} \equiv \{ f_n \mid D^{EQ}(f_n) = (\log n)^{O(1)} \}$$

What else can be done?

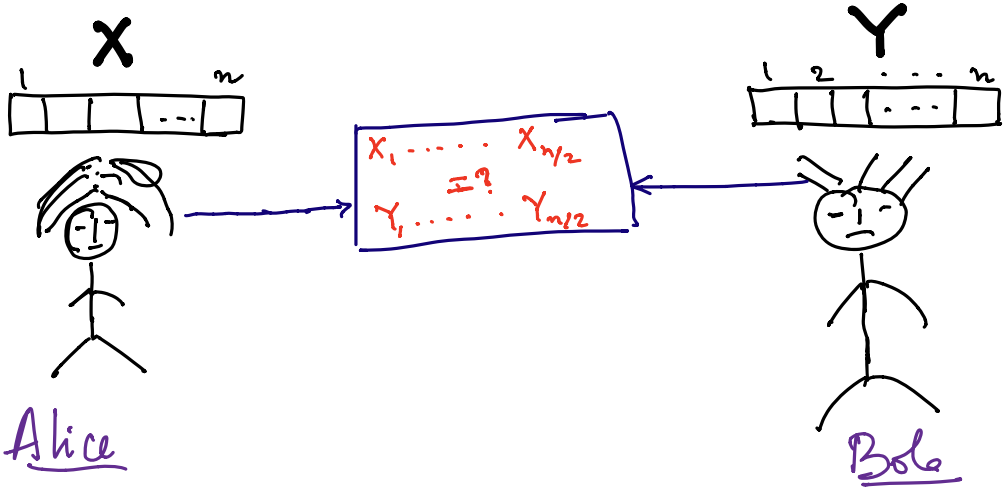
Power of EQ

$GT(X,Y) \equiv \neg(X \geq Y)$



Power Of EQ

$GT(X,Y) \equiv ?\ X \gtrless Y$



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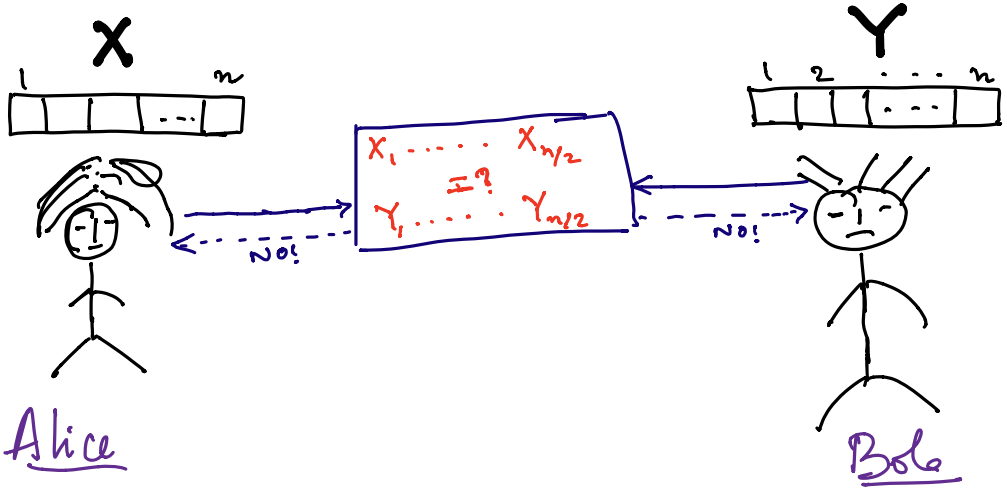

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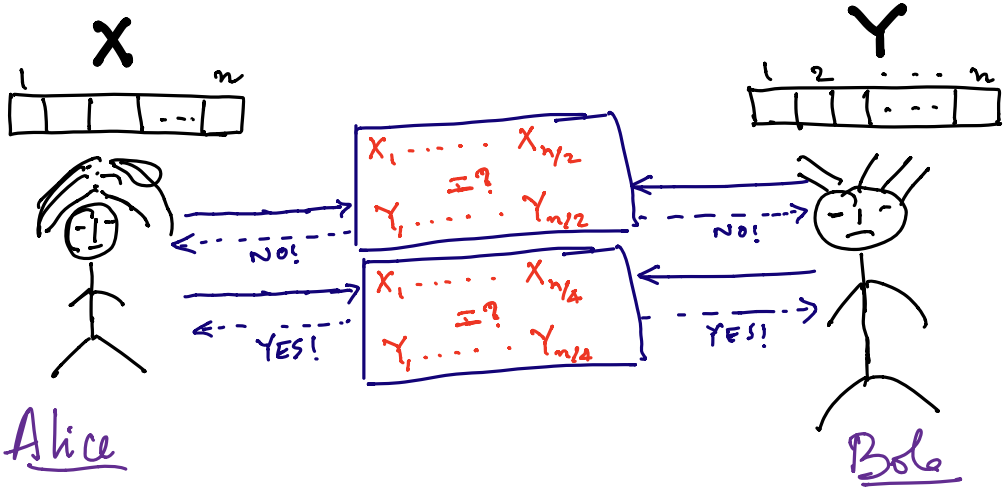

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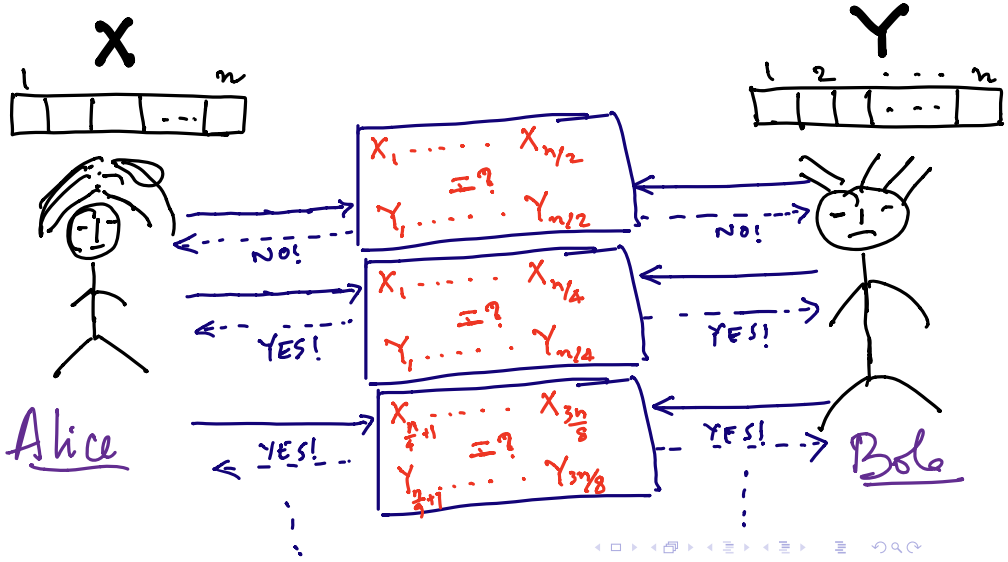
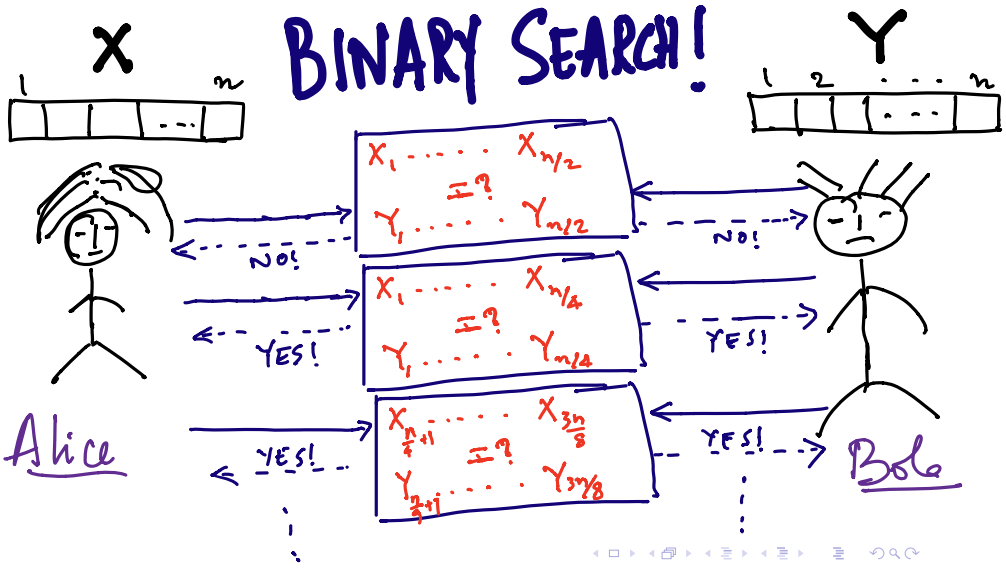


Power Of EQ

$GT(X,Y) \equiv ?\ X > Y$



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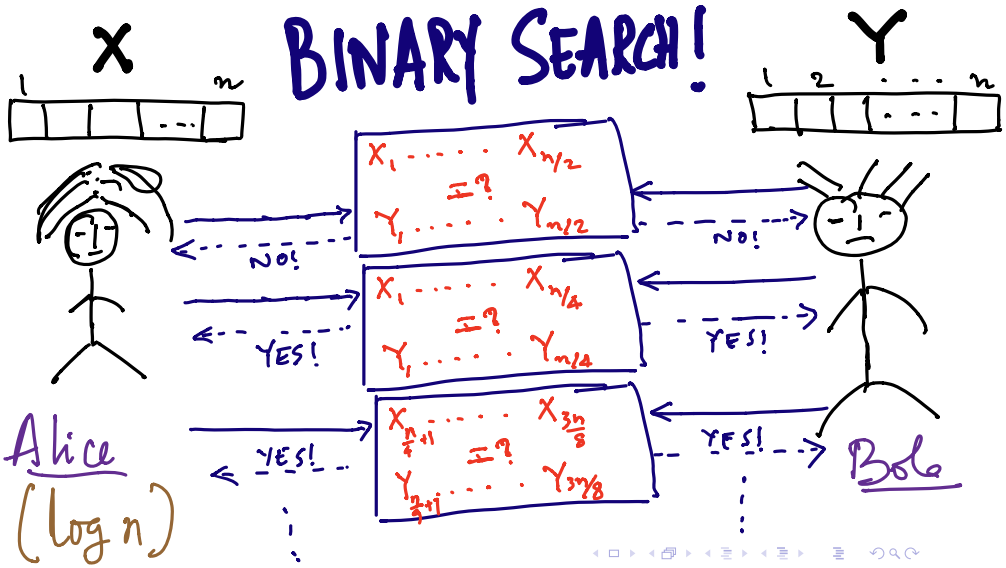

$D^{EQ}(GT) = O(\log n)$

EQUALITY vs. Randomness

$$P^{EQ} \equiv \left\{ f_n \mid D^{EQ}(f_n) = (\log n)^{O(1)} \right\}$$

Fact: $P^{EQ} \subseteq BPP$.

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Question: Is $P^{EQ} = BPP$?

Total vs. Partial

Fact: $P^{EQ} \not\subseteq BPP$. (Partial. Gap-Hamming)

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Question: Is $P^{EQ} = BPP$? (Total Functions)

Our Result

Theorem: $P^{EQ} \not\subseteq BPP$ (Total fns.)

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Theorem:

$$P^{EQ} = P^{\Pi P_2} \not\subseteq P^{\Pi P_5} \subseteq P^{\Pi P_9} \subseteq \underbrace{\cdots}_{\text{Infinite}} \subseteq BPP$$

Integer Inner Product

Let $t \in \mathbb{Z}_+$; $X_1, \ldots, X_t, Y_1, \ldots, Y_t \in \{-N, -N+1, \ldots 0, \ldots +N\}$

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Definition:

$$IIP(X_1, \dots, X_t, Y_1, \dots, Y_t) = \begin{cases} 1 \text{ if } \sum_{i=1}^{t} X_i Y_i = 0 \\ 0, \text{ Otherwise} \end{cases}$$

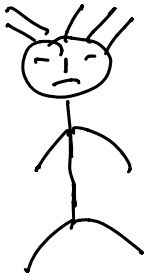
Randomized Protocol For IIP

$X_i, Y_i \in [-N, N]$

$N = 2^n$.

$X_1 \dots X_t$

$Y_1 \dots Y_t$ Y



Randomized Protocol For IIP

Random prime $p \in [n^2]$

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$X_1 \dots X_t$

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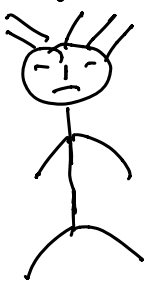

Y

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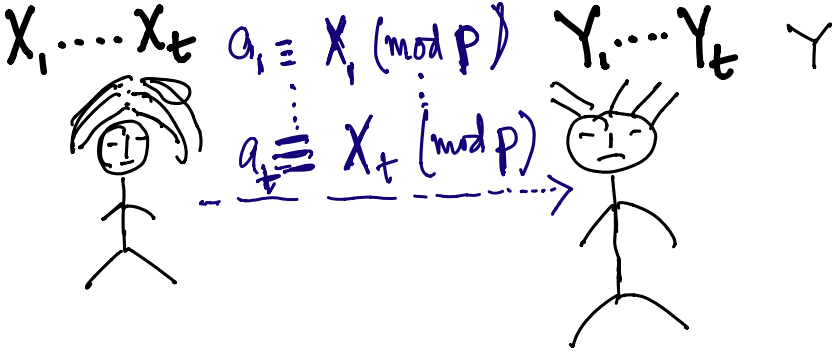

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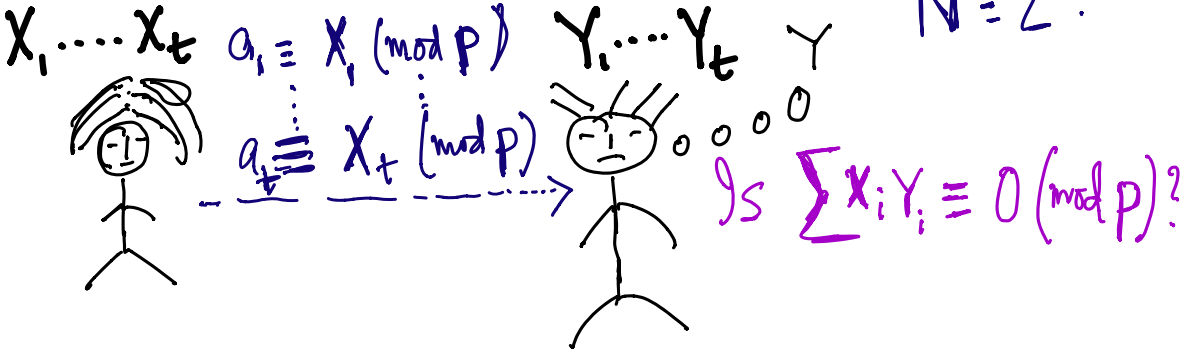

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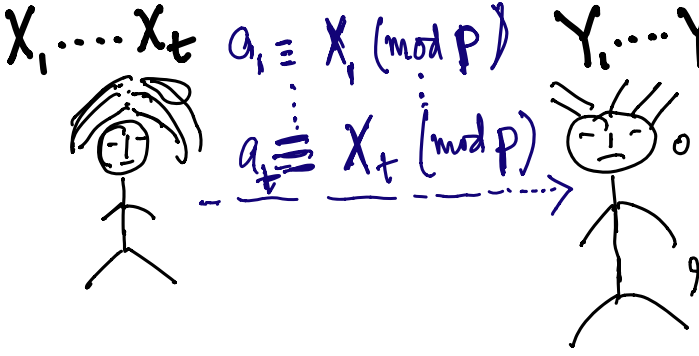

Is $\sum X_i Y_i \equiv 0 \pmod p$?

If $IIP(X,Y) \neq 0$

$\Pr_p\left[\sum X_i Y_i \neq 0 \pmod p\right] \geq 1 - o(1)$

Randomized Protocol For IIP

Random prime $p \in [n^2]$

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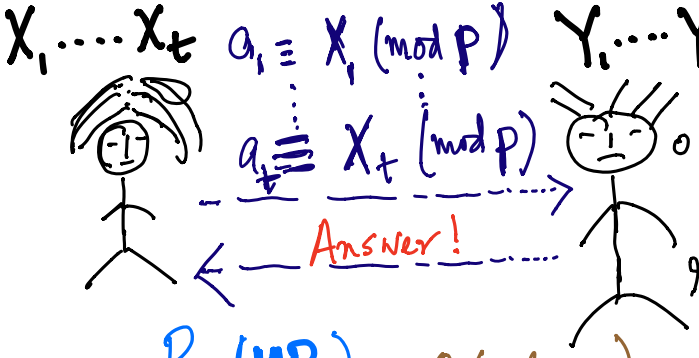


Is $\sum X_i Y_i \equiv 0 \pmod p$?

If $IIP(X,Y) \neq 0$

$R_{\frac{1}{4}}(IIP_t) = O(t \log n)$

$\Pr_p\left[\sum X_i Y_i \neq 0 \pmod p\right] \geq 1 - o(1)$

Our Result

Theorem: $D^{EQ}(\Pi P_t) = \Omega(n), t \approx 6$

Our Result

Theorem: $D^{EQ}(\|P_t) = \Omega(n)$, $t \geq 6$.

$(RP \not\subseteq P^{EQ})$

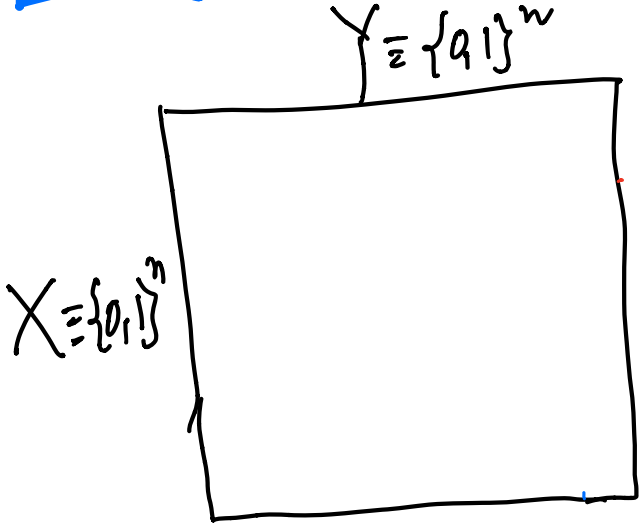
Our Result

Theorem: $D^{EQ}(\Pi P_t) = \Omega(n)$, $t \neq 6$.

$(RP \not\subseteq P^{EQ})$

Theorem: $P \subsetneq P^{EQ} \subsetneq P^{\Pi P_{t_1}} \subsetneq P^{\Pi P_{t_2}} \subsetneq \dots \subsetneq P^{\Pi P_{t_i}} \subsetneq \dots \subseteq BPP$

Lower Bounds



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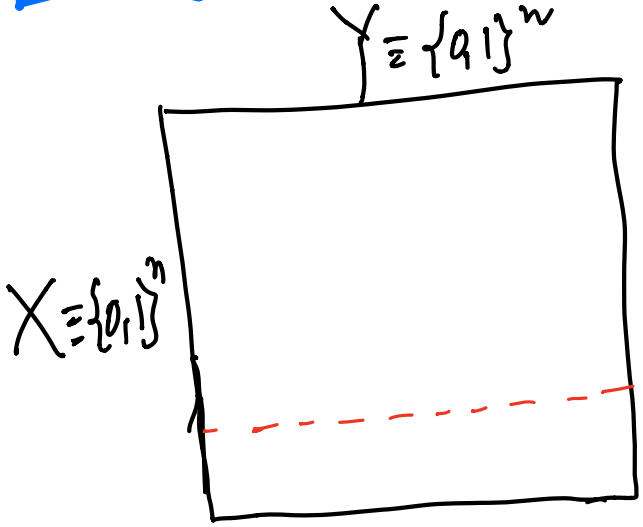

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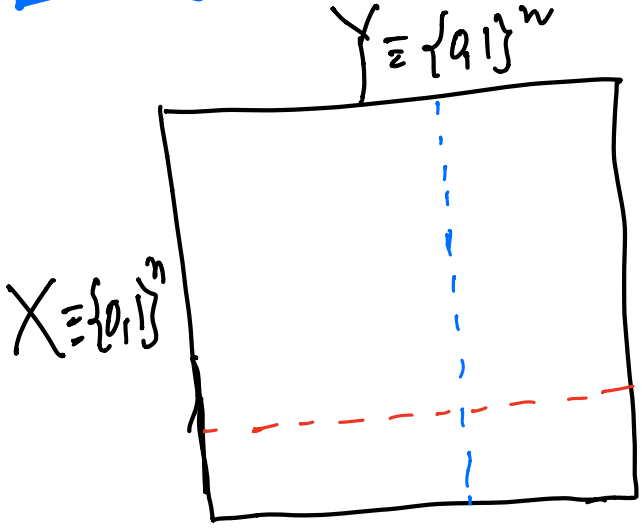

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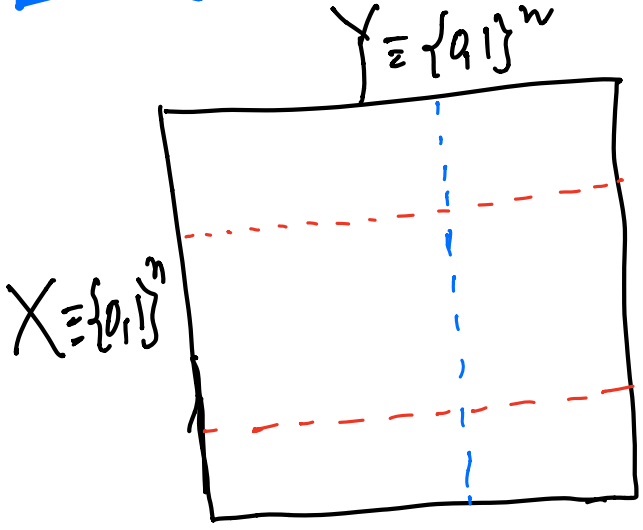

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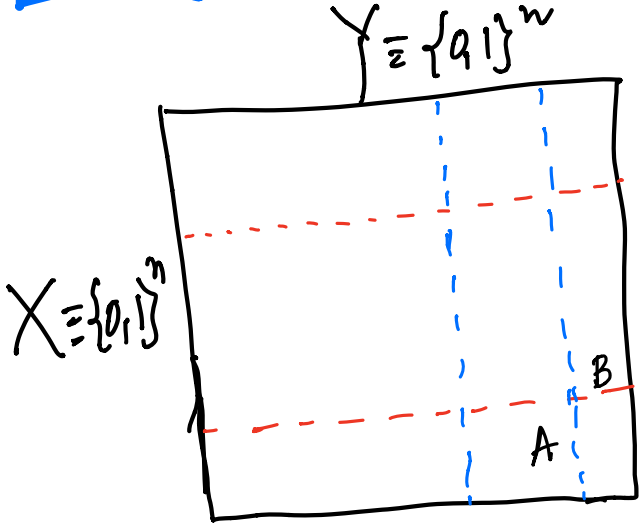

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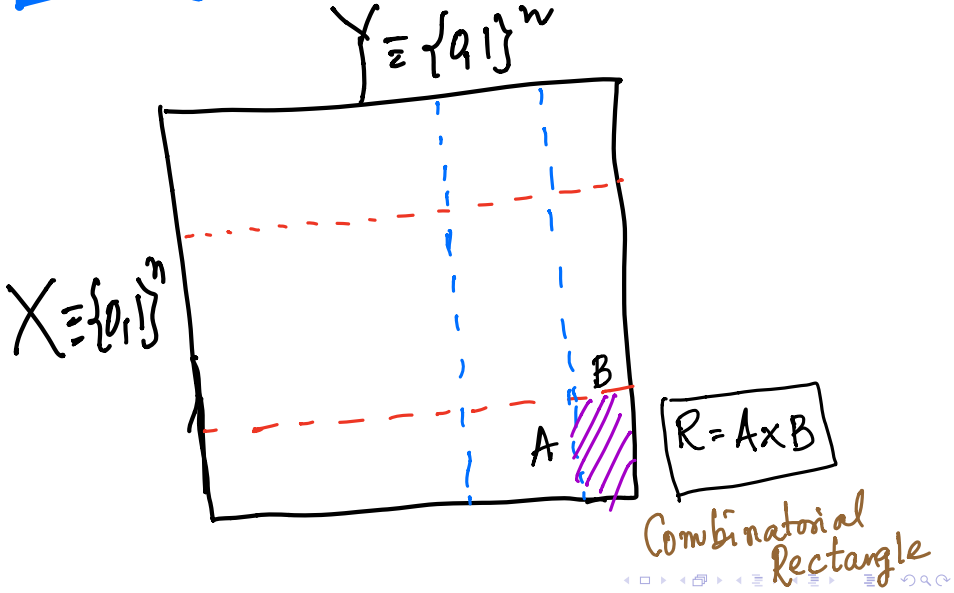

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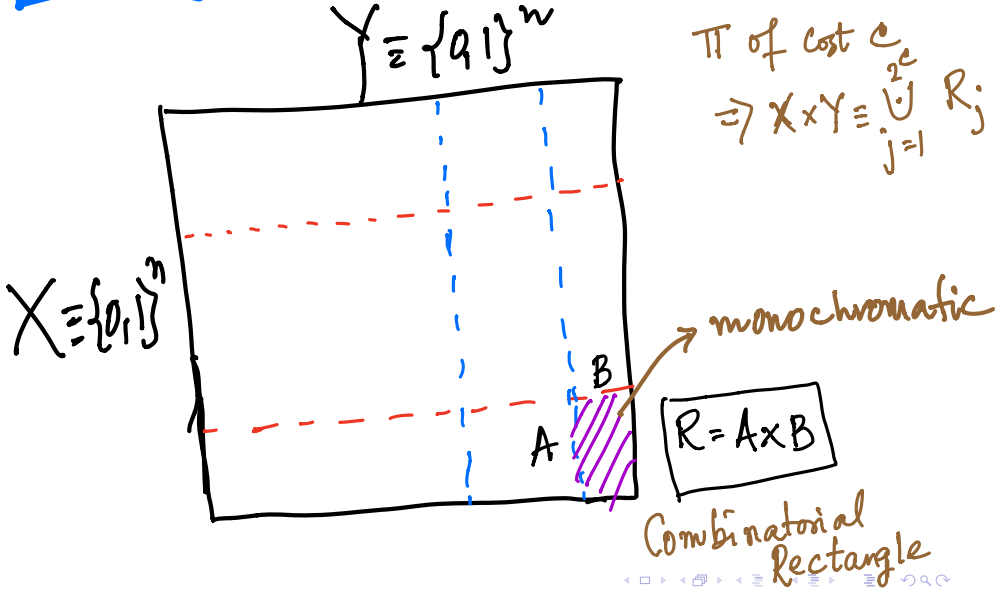
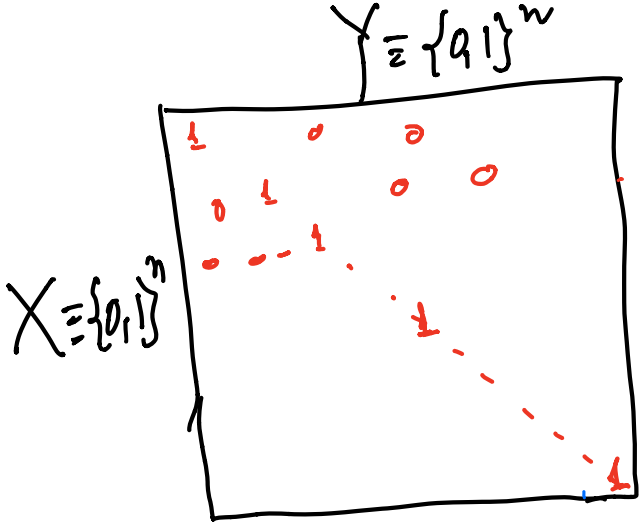


Lower Bound Fn EQUALITY



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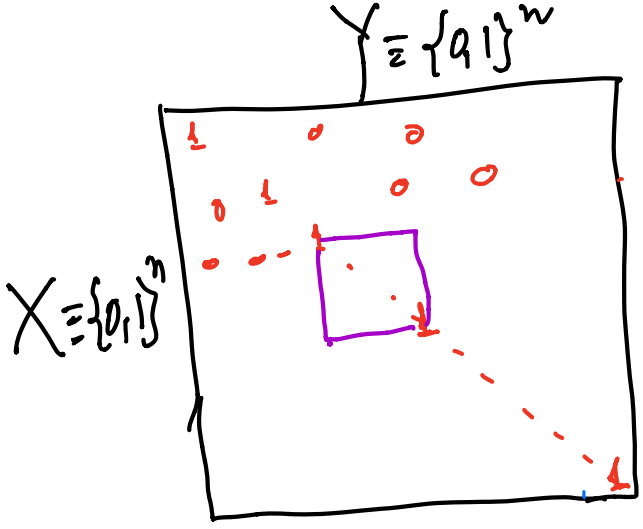

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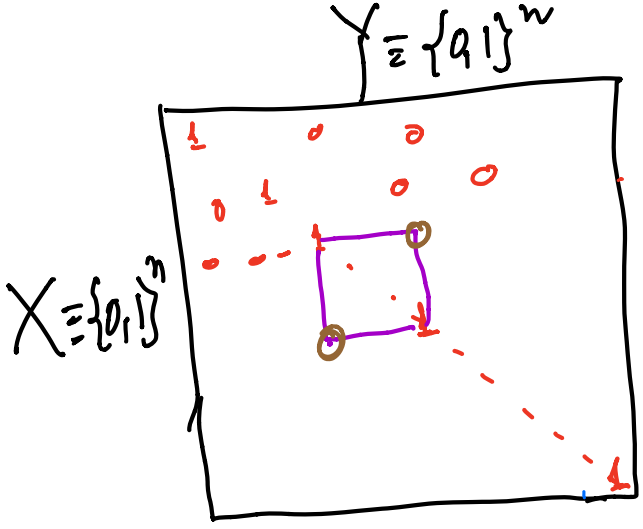

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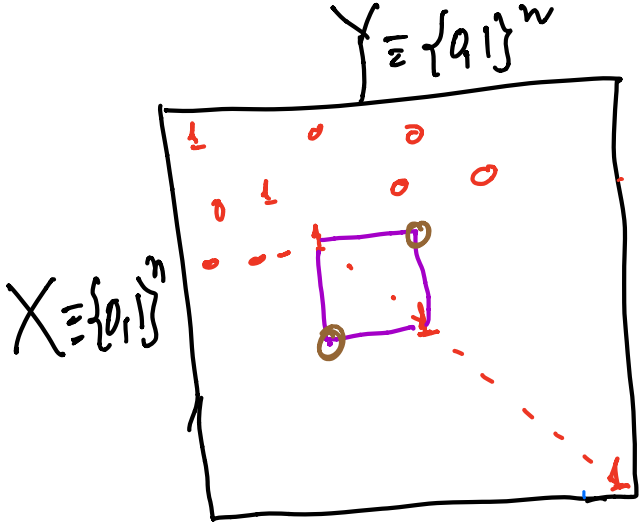
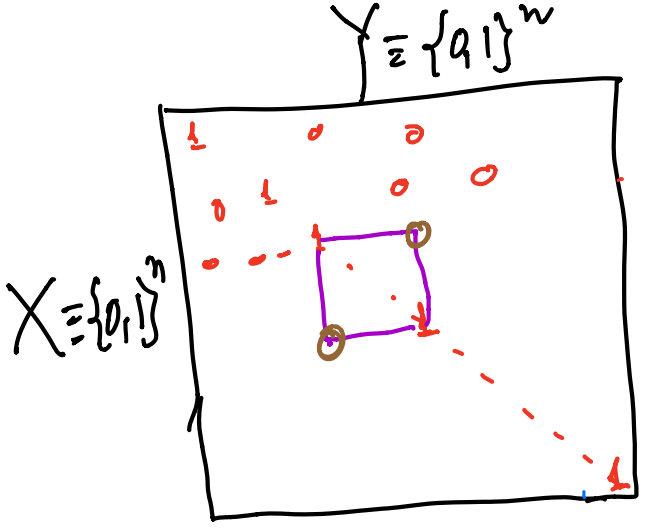


$2^c \geq 2^n + 1$

Lower Bound Fn EQUALITY



$$2^c \geq 2^n + 1$$

$$\Rightarrow c \geq n+1$$

$$\therefore \boxed{D(EQ) = n+1}$$

How Do We Lower Bound Cost of π^{EQ} Protocols?

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Consider π^{GT} protocols.

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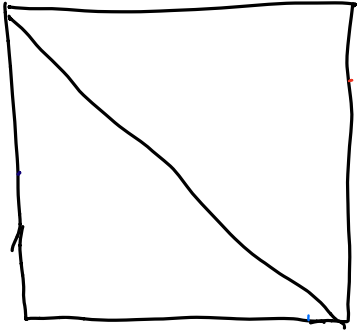
Consider π^{GT} protocols.

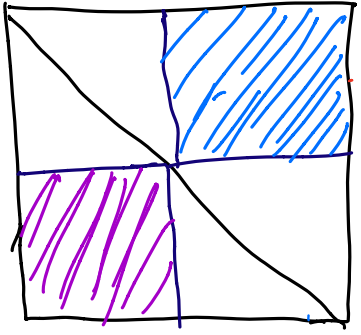
$D^{GT}(f) \leq 2 \cdot D^{EQ}(f)$.

Perimeter of EQUALITY is Small.



Difficulty:
No small partition into monochrom rectangles.

Perimeter of EQUALITY Is Small.



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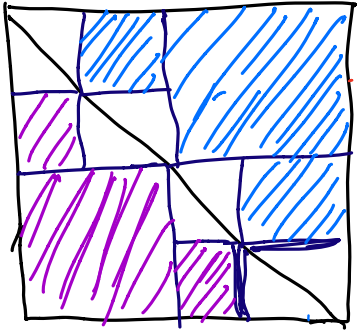
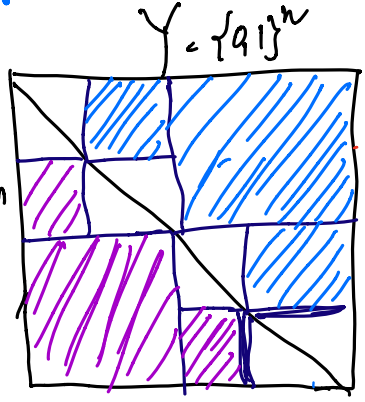


Difficulty:
No small partition into monochrom. rectangles.

Perimeter of EQUALITY Is Small.

Observation:

Average mon. rect size is large.



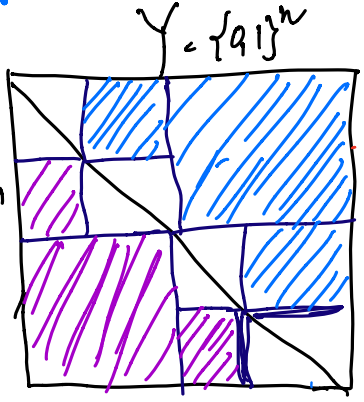
Difficulty:

No small partition into monochrom rectangles.

Perimeter of EQUALITY Is Small.

Observation:

Average mon. rect size is large. $X = \{0,1\}^n$



Difficulty:

No small partition into monochrom rectangles.

Definition: -

$R = A \times B$.

$\mathrm{Per}(R) = |A| + |B|$

$\mathcal{R} = \bigcup_i (R_i = A_i \times B_i)$

$\mathrm{Per}(\mathcal{R}) = \sum_i (|A_i| + |B_i|)$

Perimeter of EQUALITY Is Small.

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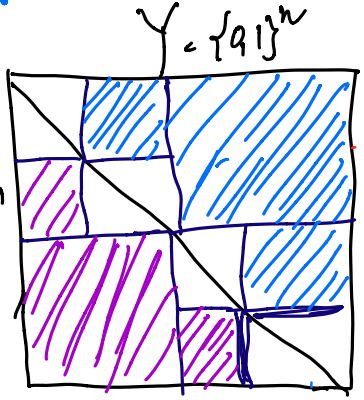

Difficulty:

No small partition into monochrom rectangles.

$$Per(EQ_n) \leq 2 \cdot (2^{n-1} + 2^{n-1}) + 2 \cdot Per(EQ_{n-1})$$

Perimeter of EQUALITY Is Small.

Observation:

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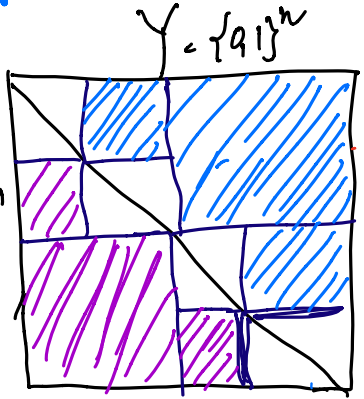
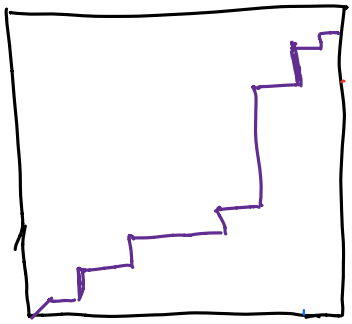


Difficulty:

No small partition into monochrom rectangles.

$$\begin{aligned} Per(EQ_n) &\leq 2\cdot(2^{n-1}+2^{n-1}) + 2\cdot Per(EQ_{n-1}) \\ &= 4\cdot 2^{n-1} + 2\cdot Per(EQ_{n-1}) \leq 4\cdot 2^{n-1} + 4\cdot 2^{n-2} + \cdots \\ &= 2n\cdot 2^{n-1} = 2n\cdot 2^{n}. \end{aligned}$$

Perimeter of Greater-than is Small.

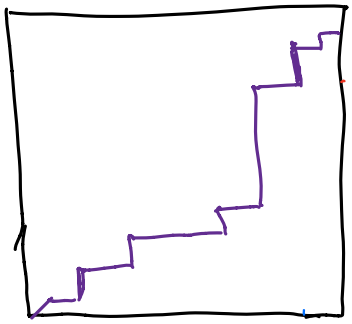


Permute rows & columns.

Perimeter of Greater-than is Small.

Monotone matrix

$i_1 \le i_2$
$j_1 \le j_2$
$\Rightarrow M(i_1, j_1) \le M(i_2, j_2)$



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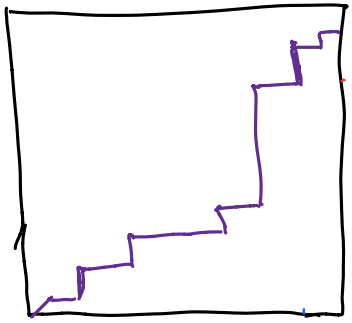

Permute rows & columns.

Fact: Let $R = A \times B$ be monotone.

$$\beta_{er}(R) \le (|A| + |B|) \log_2(|A| \cdot |B|)$$

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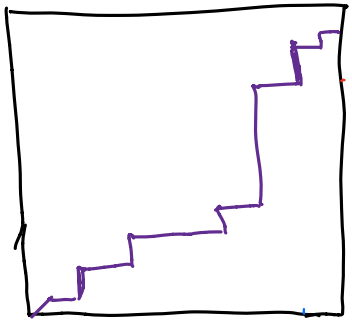

Permute rows & Columns.

Corollary:
$D^{pr}(GT_n) \le 2^{n+1} \cdot (2n)$

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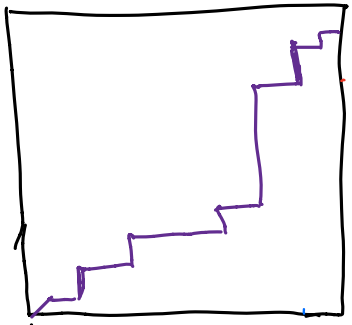

Permute rows & Columns.

Corollary:
$Per(GT_n) \le 2^{n+1} \cdot (2n)$

Fact: Let $R = A \times B$ be monotone.
$Per(R) \le (|A| + |B|) \log_2(|A| \cdot |B|)$

Lemma: Let $D^{GT}(f) \le c$. Then $Per(f) \le 2^{n+1} \cdot (2n)^c$

Functions With High Perimeter

Lemma 3.3. *Let f be an n-bit function with a corresponding $2^n \times 2^n$ communication matrix M. Assume that:*

1. *The number of entries i, j with $M_{i,j} = 1$ is $\alpha 2^{2n}$.*
2. *For any 1-monochromatic rectangle R in M it holds that $|R| \leq \beta 2^{2n}$.*

Then, $\mathrm{Per}(f) \geq 2^{n+1} \cdot \lfloor \alpha/\beta \rfloor \cdot \sqrt{\beta}$

High Perimeter

Proof: $f^{-1}(1) \equiv \mathcal{R} = \bigcup_i R_i = \bigcup_i A_i \times B_i$

$$\mathrm{Per}(\mathcal{R}) = \sum_i |A_i| + |B_i| \geq 2 \sum_i \sqrt{|A_i| |B_i|}$$

High Perimeter

Proof: $f^{-1}(1) \equiv \mathcal{R} = \bigcup_i R_i = \bigcup_i A_i \times B_i$

$$Per(\mathcal{R}) = \sum_i |A_i| + |B_i| \geq 2 \sum_i \sqrt{|A_i||B_i|}$$

$$\Rightarrow Per(\mathcal{R}) \geq 2^{n+1} \min \sum \sqrt{x_i}$$

$$x_i \equiv \frac{|A_i||B_i|}{2^{2n}}$$

$$0 \leq x_i \leq \beta$$

$$\sum x_i = \alpha$$

Lower Bounding Perimeter

$$\mathrm{Per}(R) \geq 2^{n+1} \min \sum_i \sqrt{x_i}$$

$$0 \leq x_i \leq \beta$$

$$\sum_i x_i = \alpha$$

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Concave function attains minm. on vertex of polytope.

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Vertex: $\lfloor \frac{\alpha}{\beta} \rfloor$ co-ordinates $:= \beta$.

1 co-ordinate $= \alpha - \lfloor \frac{\alpha}{\beta} \rfloor \beta$.

Lower Bounding Perimeter

$$OPT \geq 2^{n+1}\left\lfloor \frac{\alpha}{\beta} \right\rfloor \sqrt{\beta}.$$

$$Per(R) \geq 2^{n+1} \min \sum_i \sqrt{x_i}$$

$$0 \leq x_i \leq \beta$$

$$\sum_i x_i = \alpha$$

$$x_i \equiv \frac{|A_i||B_i|}{2^{2n}}$$

$$\simeq 2^{n+1}\left(\frac{\alpha}{\sqrt{\beta}}\right)$$

Concave function attains minm. on vertex of polytope.

Vertex: $\left\lfloor \frac{\alpha}{\beta} \right\rfloor$ co-ordinates $:= \beta$.

1 co-ordinate $= \alpha - \left\lfloor \frac{\alpha}{\beta} \right\rfloor \beta$.

Perimeter of $\mathbb{1}_P$ is High.

Lemma 1: Let t be even. $\Pr_{x,y}\left[\mathbb{1}_{P_{n,t}}(x,y)=1\right] = \Omega\left(\frac{1}{tN^2}\right)$

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$x = (x', -x'')$

$y = (y', y'')$

x', x'', y', y'' are i.i.d $\sim_r [-N, N]^{t/2}$

Perimeter of IIP is High.

Lemma 1: Let t be even.

$$\Pr_{x,y}\left[IIP_{n,t}(x,y)=1\right]=\Omega\left(\frac{1}{tN^{2}}\right)$$

$$x=(x',-x'')$$

$$y=(y',y'')$$

x', x'', y', y'' are i.i.d $\sim_r [-N, N]^{t/2}$

$$\langle x,y\rangle=\underbrace{\langle x',y'\rangle}-\underbrace{\langle x'',y''\rangle}$$

range: $O(tN^{2})$

Perimeter of 1P is High.

Lemma 1: Let t be even.

$$\Pr_{x,y}\left[1P_{n,t}(x,y)=1\right]=\Omega\left(\frac{1}{tN^{\nu}}\right)$$

$x = (x', -x'')$

$y = (y', y'')$

x', x'', y', y'' are i.i.d $\sim_{\nu} [-N, N]^{t/2}$

$\langle x, y\rangle = \langle x', y'\rangle - \langle x'', y''\rangle$

range: $O(tN^{\nu})$

Collision probability $= \Omega\left(\frac{1}{tN^{\nu}}\right)$

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Perimeter of $\mathbb{1}\mathbb{P}$ is High.

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Lemma 2: Rectangle $R \subseteq \mathbb{1}\mathbb{P}_{n,t}^{-1}(1) \Rightarrow |R| \leq (4N)^t$

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$R = A \times B$.

Prime P, $2N+1 \leq P \leq 4N$

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$R = A \times B$.

$A \subseteq F_p^t$, $W = \mathrm{span}(A)$ $W \perp V$

Prime p, $2M+1 \le p \le 4M$ $B \subseteq F^t$ $V = \mathrm{span}(B)$ $\Rightarrow \dim(W)+\dim(V) \le t \le (4N)^t$

Perimeter of $\mathbb{1}\mathbb{P}$ is High.

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$\alpha = \Omega\left(\frac{1}{tN^2}\right)$, $\beta \leq \frac{(4N)^t}{(2N+1)^{2t}} \leq \frac{1}{N^t}$

Perimeter of Π_P is High.

Lemma 1: Let t be even. $\Pr_{x,y}\left[\Pi_{P_{n,t}}(x,y)=1\right] = \Omega\left(\frac{1}{tN^2}\right)$

Lemma 2: Rectangle $R \subseteq \Pi_{P_{n,t}}^{-1}(1) \Rightarrow |R| \leq (4N)^t$

$\alpha = \Omega\left(\frac{1}{tN^2}\right)$, $\beta \leq \frac{1}{N^t}$

$$\text{Per}\left(\Pi_{P_{n,t}}\right) \geq N^t \cdot \left(\frac{\alpha}{\sqrt{\beta}}\right) \geq N^t \cdot N^{t/2-2}$$

D^{EQ} Cost of IIP

Let $D^{EQ}(IIP_{n,t}) = c$

$$N^t \cdot N^{t/2-2} \leq Per(IIP_{n,t}) \leq (2t \log N)^c \cdot (2N^t)$$

D^{EQ} Cost of IIP

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$$N^t \cdot N^{t/2-2} \leq Per(IIP_{n,t}) \leq (2t \log N)^c \cdot (2N^t)$$

$$\Rightarrow c \geq \Omega\left(\frac{t \log N}{\log t \log N}\right) = \Omega\left(\frac{tn}{\log n}\right).$$

Conclusion.

$RP \not\subseteq P^{EQ} \Rightarrow P^{EQ} \subsetneq BPP$

$D^{EQ}(11P_{t,n}) = \Omega\left(\frac{n}{\log n}\right)$ → improves $\Omega(n)$.

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$D^{EQ}(IIP_{t,n}) = \Omega\left(\frac{n}{\log n}\right)$ → improves $\Omega(n)$.

$D^{EQ}(DISJ_n) = \Omega(n)$.

Consequence:

TC^0 circuits need $\Omega(n)$ size for $DISJ_n$.