The Log-Approximate-Rank Conjecture is False

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20022020

Communication Complexity

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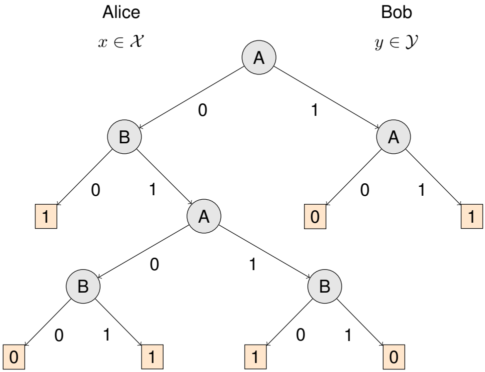
- In this talk, we focus on two parties (Alice and Bob) computing a Boolean function.

A Communication Protocol



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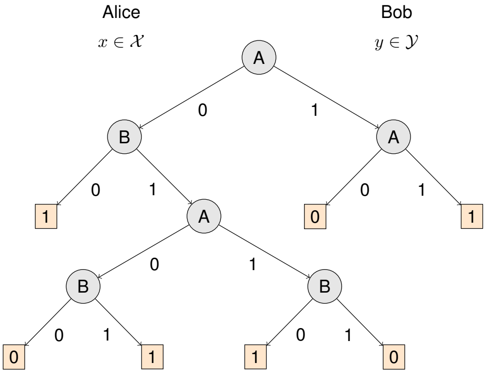

(x, y) is accepted

$\Leftrightarrow$

(x, y) reaches a 1-leaf.

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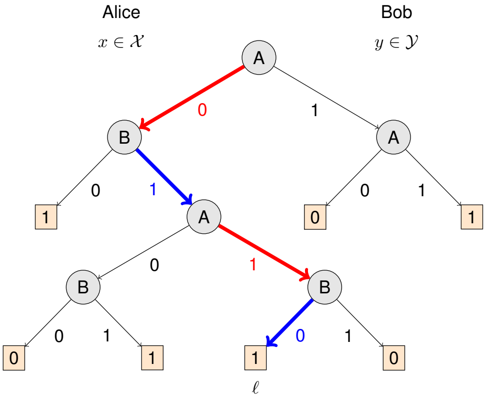

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Inputs that reach ℓ

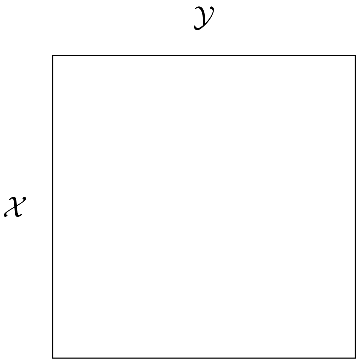
$=$

$\{x : x$ answers red$\}$

$\times$

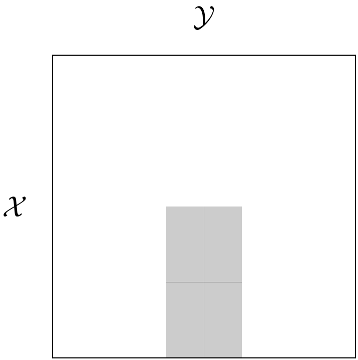
$\{y : y$ answers blue$\}$.

Rank



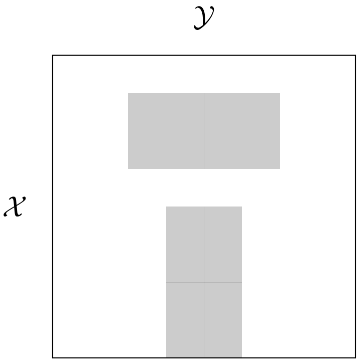
Building the truth table for the function computed by the protocol.

Rank



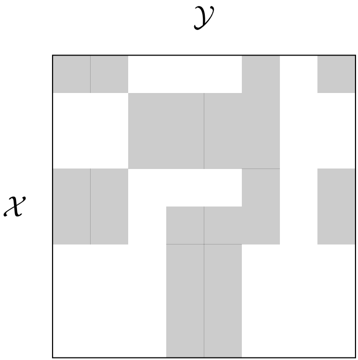
Inputs that reach leaf ℓ contribute a rank 1 matrix.

Rank



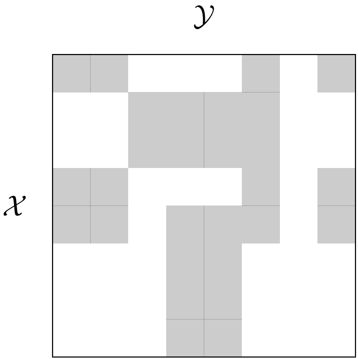
Inputs that reach leaves ℓ_1 or ℓ_2 form a rank ≤ 2 matrix.

Rank



Inputs that reach any 1 leaf form a rank $\leq 2^c$ matrix.

Rank



Cost c protocol for F

$\Longrightarrow$

M_F has rank $\leq 2^c$.

Protocol-Rank Equivalence?

Conjecture (Lovász Saks '88)

$$\exists \textit{ constant } \alpha \textit{ s.t. } D(F) \leq \log^{\alpha} \mathsf{rank}(F)$$

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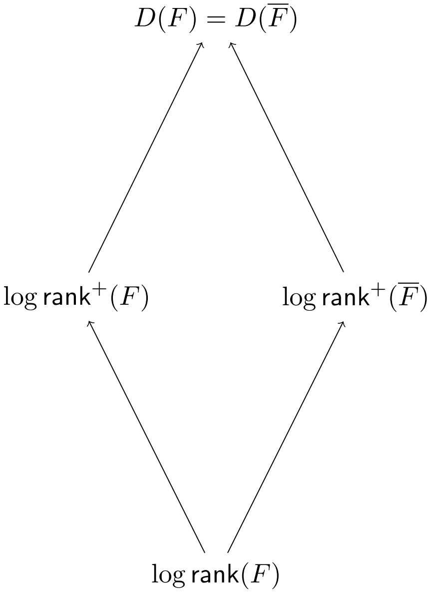
Fun fact: LRC is True if you restrict the rank decomposition to be nonnegative.

Deterministic Measures



Randomized can do better

Alice

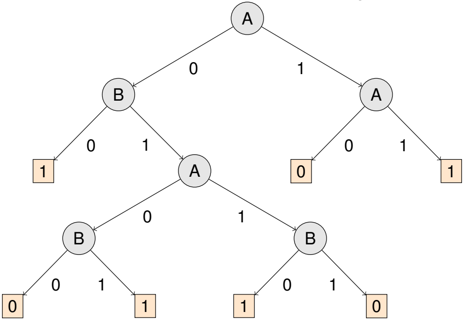
$x \in \mathcal{X}$

r_A, a string of random bits

Bob

$y \in \mathcal{Y}$

r_B, a string of random bits



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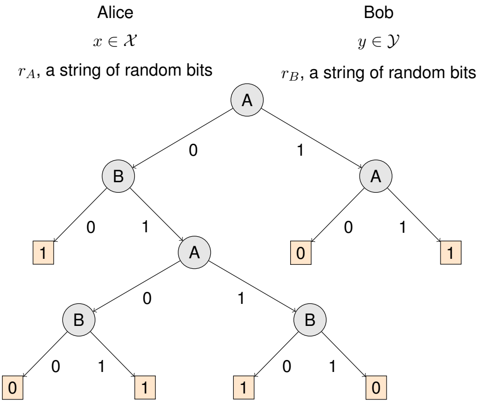

- Equality has rank 2^n. Requires n bits of communication.

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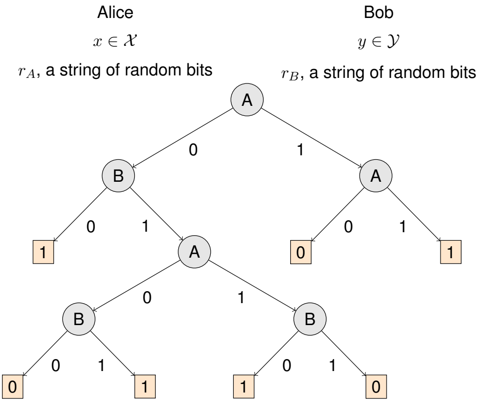

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- Equality has an $O(\log n)$ protocol. (Check equality modulo a random prime of length $O(\log n)$.)

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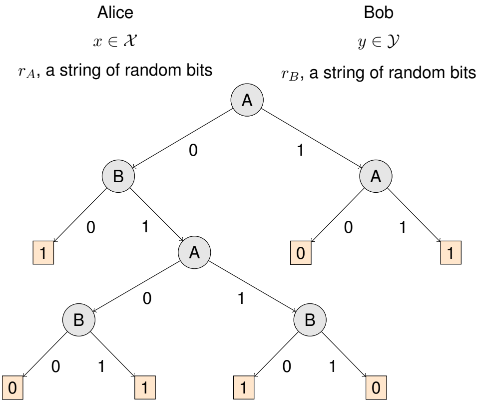

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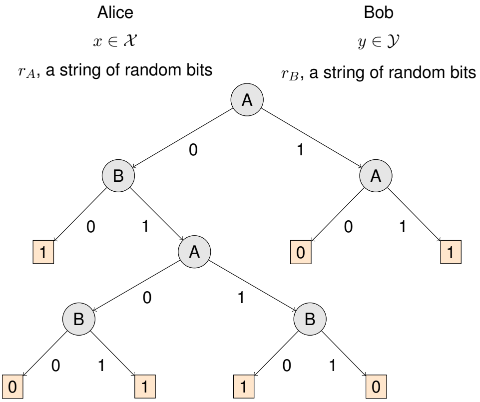

- Equality has rank 2^n. Requires n bits of communication.
- Equality has an $O(\log n)$ protocol. (Check equality modulo a random prime of length $O(\log n)$.)
- $\log$ rank doesn't serve as a lower bound anymore with randomness.
- What structure of randomized protocols can we exploit?

A Randomized Communication Protocol

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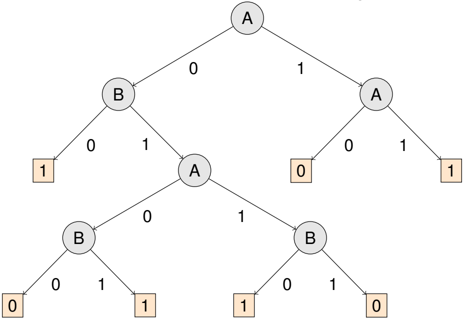
r_A, a string of random bits

Bob

$y \in \mathcal{Y}$

r_B, a string of random bits



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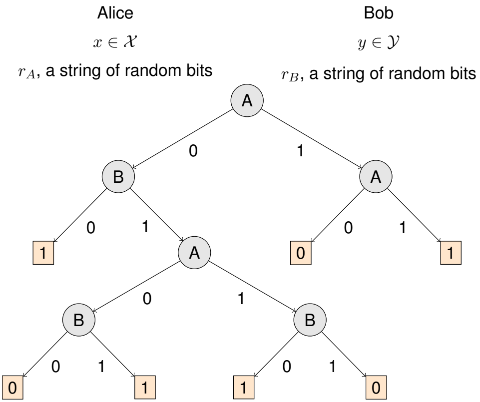
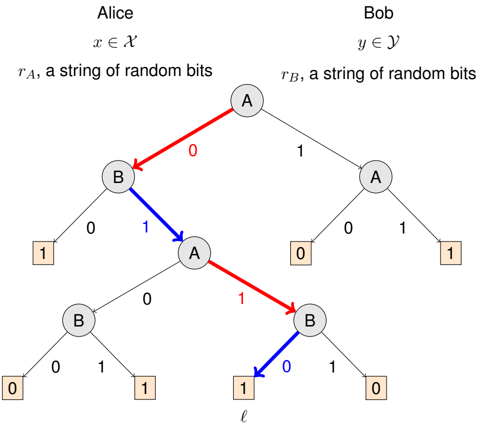


$$\Pr[(x,y) \text{ is accepted}]$$
$$=$$
$$\Pr[(x,y) \text{ reaches a } 1\text{-leaf}].$$

A Randomized Communication Protocol



$$\Pr[(x,y) \text{ is accepted}] = \Pr[(x,y) \text{ reaches a } 1\text{-leaf}].$$

$$\Pr[(x,y) \text{ reaches } \ell] = \Pr_{r_A}[x \text{ answers red}] \times \Pr_{r_B}[y \text{ answers blue}].$$

Small Approximate Rank

$\Pr_{r_B}[y \text{ answers blue}]$

$\Pr_{r_A}[x \text{ answers red}]$

	0	.5	0	.6
.5		.25		.3
.8		.4		.48
0				
0				

$\Pr[(x, y) \text{ reaches } \ell]$ is a rank 1 matrix.

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$\Pr_{r_B}[y \text{ answers blue}]$

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	0	.5	0	.6
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.8		.4		.48
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$\Pr[(x, y) \text{ reaches } \ell]$ is a rank 1 matrix.

$\Pr[(x, y) \text{ is accepted}]$ is a rank $\leq 2^c$ matrix.

$$\boxed{\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}$$

M_F

$$\boxed{\begin{matrix} .8 & .9 & .1 & .2 \\ 0 & .9 & .1 & .1 \\ 0 & .1 & .8 & 0 \\ .1 & 0 & 0 & 1 \end{matrix}}$$

$M_{\text{Pr of accepting}}$

1	1	0	0
0	1	0	0
0	0	1	0
0	0	0	1

M_F

$.8$	$.9$	$.1$	$.2$
0	$.9$	$.1$	$.1$
0	$.1$	$.8$	0
$.1$	0	0	1

$M_{\text{Pr of accepting}}$

Rank $\leq 2^c$

1	1	0	0
0	1	0	0
0	0	1	0
0	0	0	1

M_F

Approx. Rank $\leq 2^c$

.8	.9	.1	.2
0	.9	.1	.1
0	.1	.8	0
.1	0	0	1

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$M_{\text{Pr of accepting}}$

Rank $\leq 2^c$

$$\log \mathsf{rank}_{1/3}(F) \leq c.$$

Protocol-Rank Equivalence?

Conjecture (ForgeGod '05, Lee Shraibman '07)

$\exists$ *constant* β *s.t.* $R(F) \leq \log^{\beta} \mathsf{rank}_{1/3}(F)$

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For a randomized protocol, the number of bits exchanged in the worst case, R(f), is conjectured to be polynomially related to the following absurd formula:

min{rank(M'_f): $M'_f \in \mathbb{R}^{2^n \times 2^n}$, $(M_f - M'_f)_\infty \leq 1/3$}.

Figure: Screenshot from "Communication complexity - Wikipedia" (Dec '05)

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Set Disjointness shows that $\beta \geq 2$. [Kalyanasundaram Schnitger '92, Razborov '92]

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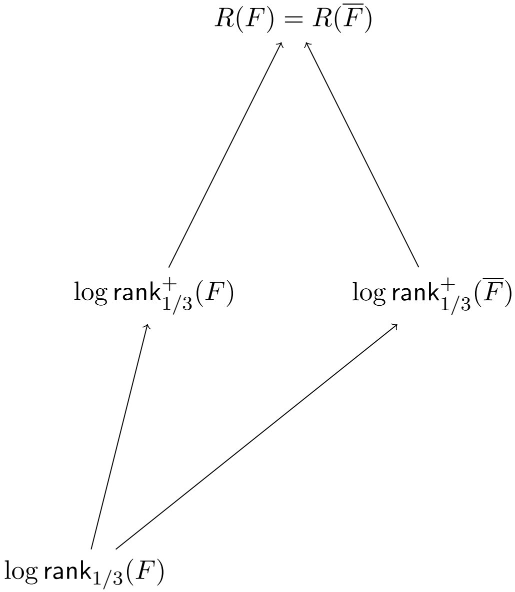
[Göös Jayram Pitassi Watson '17] showed that $\beta \geq 4$.

Randomized Measures



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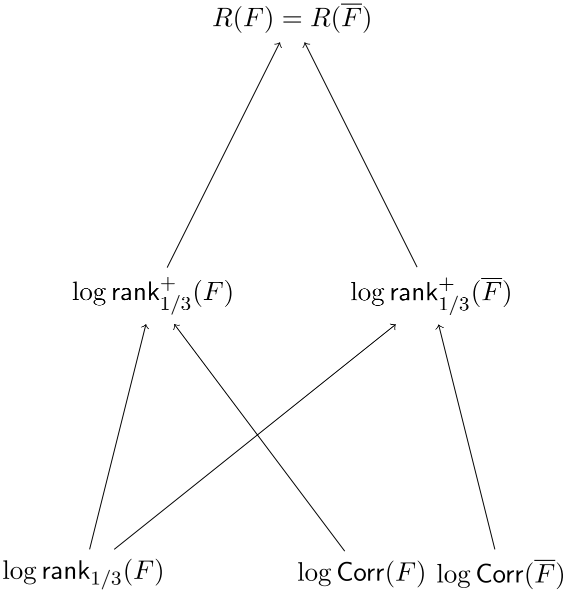
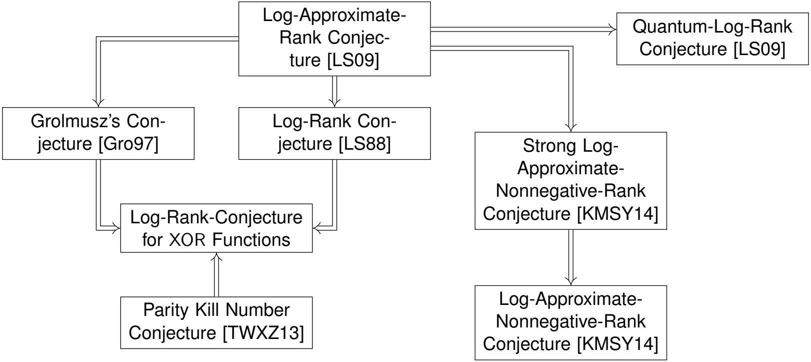


The Web of Conjectures



Protocol-Rank Non-Equivalence

Theorem (Chattopadhyay Mande S '19)

There is a function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *such that* $\mathsf{rank}_{1/3}(F) \leq O(n^2)$*, but* $R(F) \geq \Omega(\sqrt{n})$*.*

Protocol-Rank Non-Equivalence

Theorem (Chattopadhyay Mande S '19)

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The Function

$$\mathsf{SINK} : \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$

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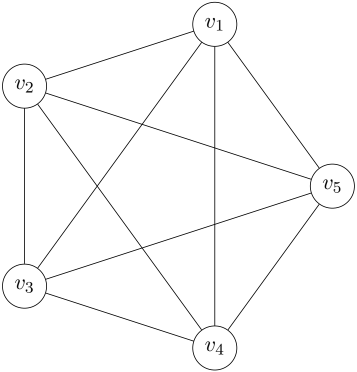

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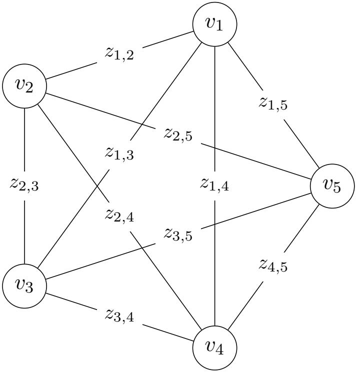

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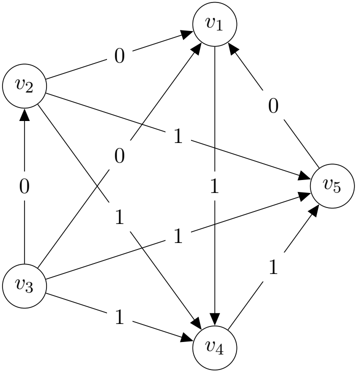

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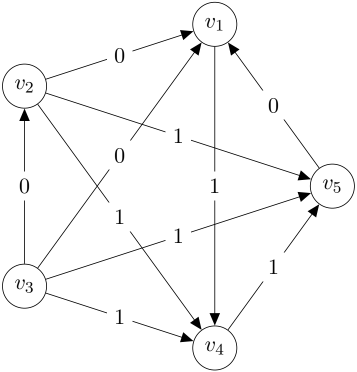
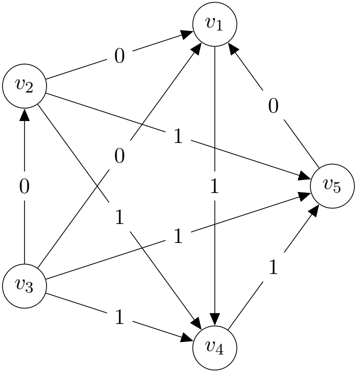


$\mathsf{SINK}(z) = 1$ iff there is a sink in the graph G_z.

The Function

$$F := \mathsf{SINK} \circ \mathsf{XOR} : \{0,1\}^{\binom{m}{2}} \times \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$



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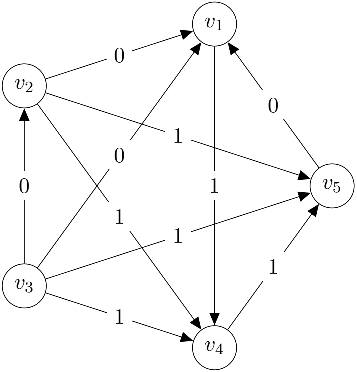

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Alice

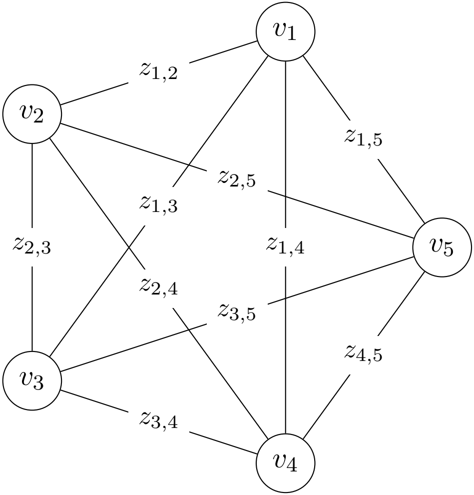
$x \in \{0,1\}^{\binom{m}{2}}$

$z = x \oplus y$

Bob

$y \in \{0,1\}^{\binom{m}{2}}$

Small Approximate Rank



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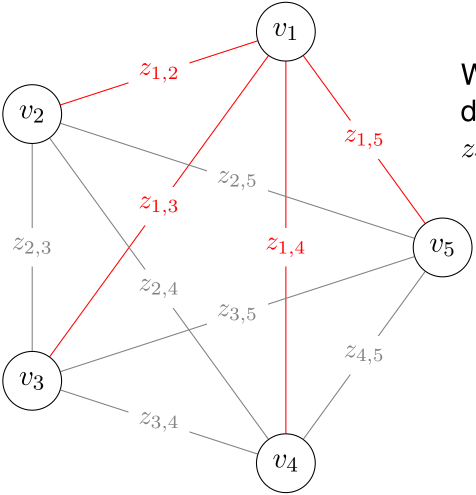

Whether or not v_1 is a sink is decided by the red variables, z_{v_1}.

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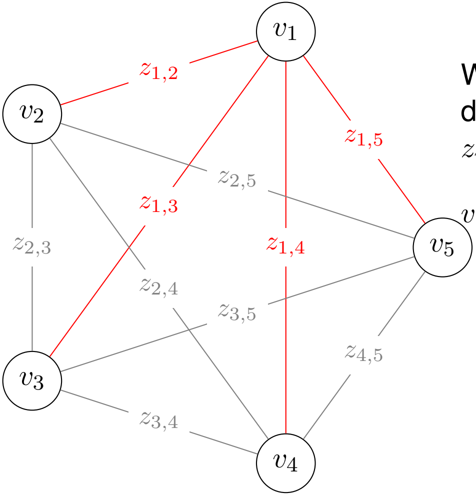

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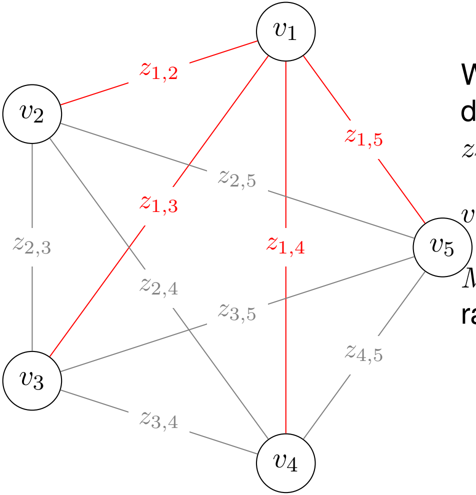

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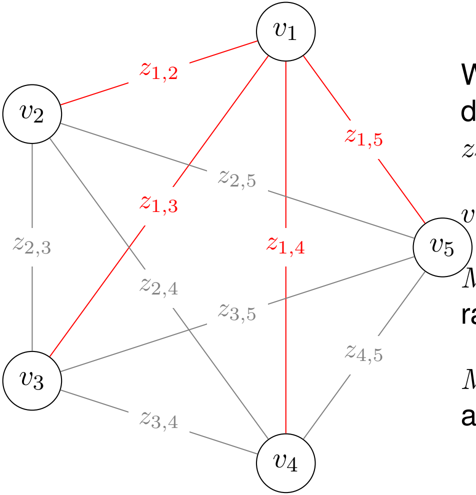
$M_{v_1 \text{ is a sink}}$ has approximate rank $\leq 2^{R(EQ)} = m^{O(1)}$.

Small Approximate Rank



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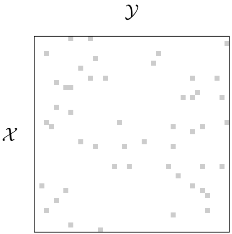
$M_F = \sum M_{v_i \text{ is a sink}}$ has approximate rank $m^{O(1)}$.

Randomized Communication Lower Bound

Assume we have a cost c randomized protocol solving SINK with error $\leq .01$.



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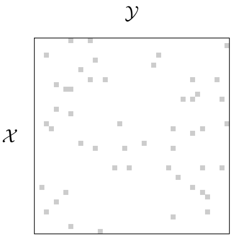

$$\Pr[\text{input has a sink}] = \frac{m}{2^{m-1}}$$

Randomized Communication Lower Bound

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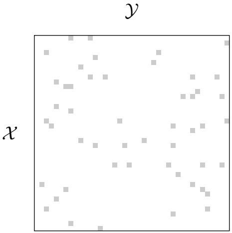
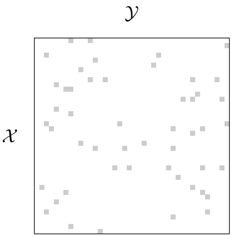


$\mathbb{E}[$ # of non-sinks rejected$] \geq .99$ # of non-sinks

Randomized Communication Lower Bound

Assume we have a cost c randomized protocol solving SINK with error $\leq .01$.

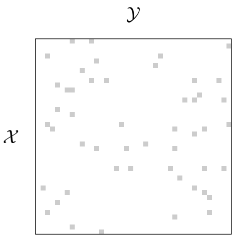


$\mathbb{E}[\text{\# of sinks rejected}] \leq .01\text{\# of sinks}$

Randomized Communication Lower Bound

Assume we have a cost c randomized protocol solving SINK with error $\leq .01$.

$$\frac{\mathbb{E}[\text{\# sinks rejected}]}{\mathbb{E}[\text{\# non-sinks rejected}]} \leq \frac{1}{99} \frac{\text{\# sinks}}{\text{\# non-sinks}}$$



Randomized Communication Lower Bound

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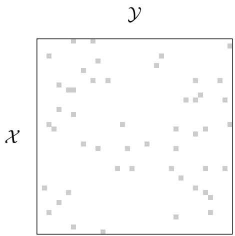
$$\frac{\mathbb{E}[\text{\# sinks rejected}]}{\mathbb{E}[\text{\# non-sinks rejected}]} \leq \frac{1}{99} \frac{\text{\# sinks}}{\text{\# non-sinks}}$$



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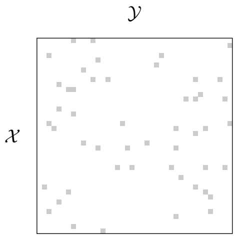

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There is a rectangle R such that $|R| \geq \Omega(2^{2n-c})$ and

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- In terms of entropy, $H(\mathcal{A}_{v_1}\mathcal{B}_{v_1}) \leq 2(m-1) - 2$.
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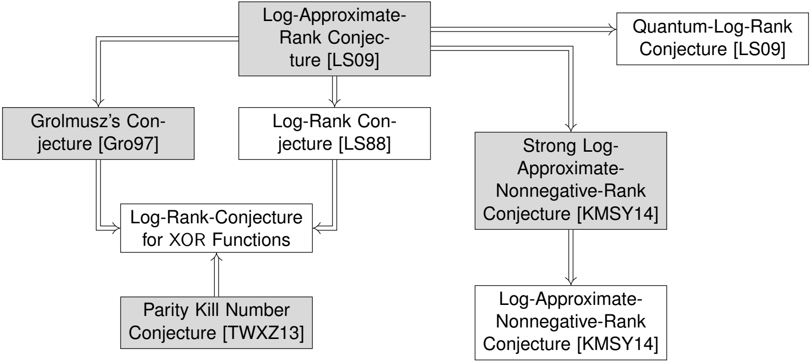
- Using this as before, we get $H(\mathcal{A}\mathcal{B}) \leq \frac{4\binom{m}{2} - 2m/3 \cdot \Omega(1)}{2} = 2\binom{m}{2} - \Omega(m)$.

Other Sunken Conjectures



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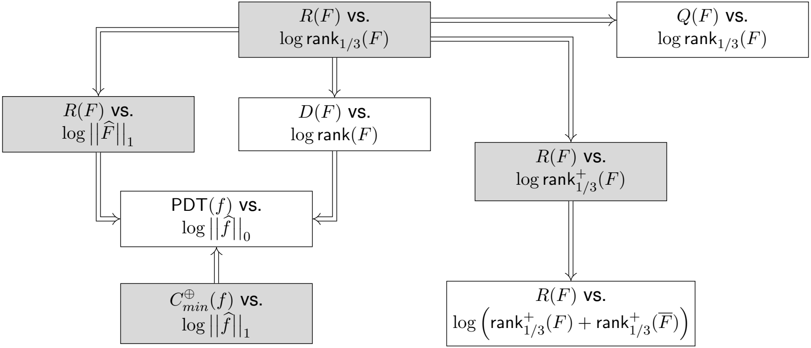

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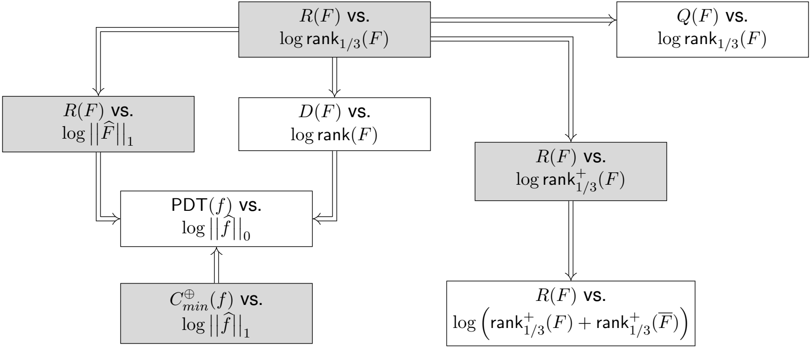

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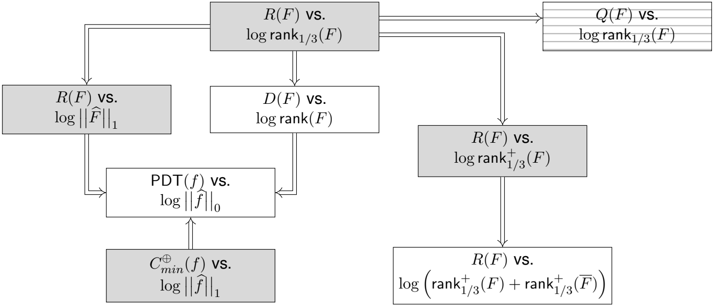
[Anshu Boddu Touchette '18, Sinha & de Wolf '18]



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So what now?

- ~~Quantum vs Log Approximate Rank?~~
- Can the Log Approximate Nonnegative Rank Conjecture be similarly refuted?
- What other functions refute the LARC?

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  Answer: **No.** [Ehrenfeucht Haussler '89]

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  Answer: **No.** [Ehrenfeucht Haussler '89]
- What if we looked at affine subspaces instead of subcubes?

If I somehow still have time

Skipped proofs:

- If $\Pr[\mathcal{A}_{v_1} = \mathcal{B}_{v_1}] \leq .1 \cdot 2^{-(m-1)}$, then $H(\mathcal{A}_{v_1}\mathcal{B}_{v_1}) \leq 2(m-1) - \Omega(1)$.
- If $f^{-1}(0)$ and $f^{-1}(1)$ are disjoint unions of few subcubes, then $f \circ$ XOR is easy for randomized communication.

Open Questions

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Thank You. Questions are welcome.

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