

Assignment 3

Discrete Mathematics - MTech CS 2018

1. Let $x > -1$ be a real number. Prove that $(1 + x)^n \geq 1 + nx$ for all natural numbers n .
2. Prove that $\sum_{i=1}^n i \times i! = (n + 1)! - 1$.
3. Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n . Prove that $a_n = 2^n + 3^n$ for all natural number n .
4. For natural number p and q , the Ramsey number $R(p, q)$ is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who don't know each other. Prove that Note that $R(p, 1) = R(1, q) = 1$. Prove that:
 - (a) $R(p + 1, q + 1) \leq R(p, q + 1) + R(p + 1, q)$
 - (b) $R(p, q) \leq C_{p-1}^{p+q-2}$
5. In a party there are $2n$ participants, where n is a natural number. Some participants shake hands with other participants. It is known that there do not exist three participants who have shaken hands with each other. Prove that the total number of handshakes in not more than n^2 .
6. The Fibonacci sequence is defined as $x_0 = 0$, $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ for all non-negative integers n . Prove that
 - (a) $x_m = x_{r+1}x_{m-r} + x_r x_{m-r-1}$ for all integers $m \geq 1$ and $0 \leq r \leq m - 1$;
 - (b) x_d divides x_{kd} for all positive integers d and k .
7. A graph $G = (V, E)$ is k colorable if there exist a function from $c : V \rightarrow \{1, 2, \dots, k\}$ such that if $(i, j) \in E$ then $c(i) \neq c(j)$. It is like can we color the vertices with k colors such that no two adjacent vertices has the same color.
 - (a) Prove that if a graph has maximum degree less than or equal to k then it is $(k + 1)$ -colorable.
 - (b) Prove that if all vertices in G has degree ≤ 3 and at least one vertex has degree < 3 then the graph is 3-colorable.
8. Prove that a graph is bipartite if and only if the graph has no odd cycle.

9. If G is a graph such that all vertices has degree more that 2 then G has a cycle.
10. If G is a graph such that all the vertices has even degree then prove that G can be written as a union of edge-disjoint cycles.
11. A simple undirected graph where any two vertices are connected is called a **connected graph**. A simple undirected graph which has no cycle is called **acyclic**. An acyclic connected graph is called a tree.
- Prove that every tree has atleast 2 vertex of degree 1.
 - Prove that a tree on n vertices has exactly $(n - 1)$ edges.
 - Prove that is a graph G is connected and has exactly $(n - 1)$ edges then G is a tree.
 - A tree where every vertex has degree 3 or 1 (except the root that has degree 2 and degree zero is the tree is of size 1) is called a binary tree. Prove that a binary tree on n vertices has at least $\lfloor n/2 \rfloor + 1$ number of leaves. (Hint: induction pivoting at the root).
12. The following are three wrong statments and their proofs. Find out where the mistake is in each of the three case. (10 Marks each)
- Statement: All cows are the same colour.
Proof. We show by induction that in any collection of n cows all n of them are the same colour.
 Base Case: If $n = 1$ the therom holds.
 Inductive Step: For the inductive step assume that in any collection of n cows all n of them are the same colour. Consider a set $n + 1$ of cows numbered 1 to $n + 1$. By the induction hypothesis cows 1 to n are the same colour and similary cows 2 to $n + 1$ are the same colopur. But the middle cows 2 to n cant change colour according to who they are grouped with so cows 1 to $n + 1$ must all be the same colour. ■
 - Statement: For a list of length n , mergesort takes $O(n)$ time.
Proof. Let $P(n)$ denote the number of steps taken by merge sort to sort n items.
 Base Case: merge sort on the empty list just returns the empty list. So $P(0) = 0$.
 Strong induction: Assume $P(1), \dots, P(n - 1)$ and try to prove $P(n)$. We know that at each step in a recursive mergesort, two approximately "half-lists" are mergesorted and then "zipped up". The mergesorting of each half list takes, by induction, $O(n/2)$ time. The zipping up takes $O(n)$ time. So the algorithm has a recurrence relation of $M(n) = 2M(n/2) + O(n)$ which is $2O(n/2) + O(n)$ which is $O(n)$. ■
 - Statement: Every graph with more than 3 vertices and minimum degree 2 contains a 3-cycle.
Proof. We proceed by induction on $|V(G)|$. As a base case, observe that the theorem is true when $|V(G)| = 3$, since any simple graph on three vertices with all vertices of degree ≥ 2 must be a cycle of length 3.
 To prove the inductive step, let G be a graph on $n - 1$ vertices for which the theorem holds, and construct a new graph G_0 on n vertices by adding one new vertex to G and

≥ 2 edges incident with this new vertex. Since G contained a cycle of length 3, the graph G_0 also contains a cycle of length 3. This completes the proof. ■