Assignment 5
Discrete Mathematics - MTech CS 2018

All the problems marked with (*) are a bit hard and may need ideas not necessarily cover in the class so far. But you are encouraged to try the problems before the sollutions are discussed in class.

1. How many ways can you divide n identical chocolates among k children such that each gets at least 2?
2. How many shortest paths are there $(0, 0)$ to (n, k) in the square integer grid?
3. Give the closed form expression for $\sum_{k=0}^{n} \binom{n}{k} (2/3)^k$.
4. Count the number of increasing (and non-increasing) functions from $[k]$ to $[n]$.
5. Count the strings of length n over the alphabet $\{A, B\}$ without consecutive occurrences of A. (For example $ABBA$ counts but not $BAAB$).
6. Count the solutions of the equation $x_1 + x_2 + \cdots + x_k = n$, where each x_i is an integer ≥ 2.
7. In how many ways can five distinct books be tied up in at most three bundles? (Here the order of books in the bundle does not matter and the bundles are not to be distinguished from one another.)
8. In how many ways can five distinct balls be distributed among three persons, so that each person has at least one ball?
9. If G is a labeled complete graph, K_n, on n vertices and u, v, w be three distinct vertices in the vertex set of G. How many different paths are there from u to v passing through w?
10. Show that $\sum_{k=1}^{n} k\binom{n}{k} 2^{k-1} 2^{n-k} = n4^{n-1}$.
11. How many ways are there to distribute 12 indistinguishable balls into 8 distinguishable cells?
12. Let $S(n, k)$ denote the number of ways of partitioning n into k parts. Prove that

$$S(n, k) = kS(n-1, k) + S(n-1, k-1).$$

13. Count the number of 0/1-strings of length n that contains at least $(n-3)$ consecutive 1's.

14. How many eight digit numbers are there that contains a 8 and 69?

15. Coefficient of $\frac{1}{x}$ and $\frac{1}{x^2}$ in $\left(x + \frac{1}{x}\right)^n$.

16. How many functions are there from $\{1, \dots, n\}$ to $\{1, \dots, m\}$?

17. Combinatorial proof of $\sum \binom{n}{k}^2 = \binom{2n}{n}$.

18. Closed form of $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$

19. Count the number of 20 digit integers in which no two consecutive digits are same.

20. If A and B are two sets such that $|A| = n$ and $|B| = k$ then how many distinct subsets of A are there whose intersection with B has size 1?

21. 20 people are sitting around a table. How many ways can we choose 3 person such that no two of the chosen ones are neighbors?

22. Prove that $1 + \binom{n}{1}2 + \binom{n}{2}4 + \dots + \binom{n}{n}2^n = 3^n$. Give a combinatorial proof.

23. Give a combinatorial proof of the following:

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}.$$

24. A composition of n is an expression of n as an ordered sum of positive integers. For example: $4 = 1 + 3$ and $4 = 3 + 1$ are different. Fix k, n. Show that $\sum c_1 \dots c_k = \binom{n+k-1}{2k-1}$ where the sum ranges over all compositions $c_1 + c_2 + \dots + c_k$ of n into k parts.

25. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_k\}$. Count the number of surjections (onto maps) from X to Y.

26. Let G be a simple path of length n. A valid coloring of the path is an assignment of colors to the vertices such that no edge is monochromatic (ie. has both end points of the same color). The goal is to compute the number of ways to color the path with five colors (red, green, blue, yellow, violet) in three different scenarios:

    (a) There are no more restrictions.

    (b) For every color there is at least one node colored with that color.

    (c) The colors red and green do not appear one next to the other. (Note that the constraint in the previous part does not apply).

27. Prove that for every $k \geq 3$ there is a number N(k) such that for every set $|S| \geq N$ of points on the plane there is a subset $T \subset S$ of size $|T| = k$, such that all points in T lie on line, or no three points in T lie on the same line.