Discrete Mathematics 9/10/2018

Lecture 1: Assignment 5 Part-2

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10. Show that $\sum_{k=1}^{n} k\binom{n}{k}2^{k-1}2^{n-k} = n4^{n-1}$.

Answer:

$2^{n-1}\sum_{k=1}^{n} k\binom{n}{k}$ —-(1)

$(1+x)^n = \binom{n}{0}x + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

differentiate with respect to x

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + 4\binom{n}{4}x^3 + ... + (n-1)\binom{n}{n-1}x^{n-2} + n\binom{n}{n}x^{n-1}$

put x = 1

$n(1+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + ... + (n-1)\binom{n}{n-1} + n\binom{n}{n}$

$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$ —-(2)

put value of eq(2) in eq(1)

$n2^{n-1}2^{n-1} = n4^{n-1}$

11.How many ways are there to distribute 12 indistinguishable balls into 8 distinguishable cells ?

Answer:

let n indistinguishable balls and k distinguishable cells

for empty cells : $\binom{n+k-1}{k-1}$

$\binom{12+8-1}{7}$

atleast one ball in cell so give one ball to k cells

for non-empty cells : $\binom{n-k+k-1}{k-1}$

$\binom{12-8+8-1}{7}$

12. Let S(n, k) denote the number of ways of partitioning n into k parts. Prove that S(n, k) = kS(n - 1, k) + S(n - 1, k - 1).

Answer:

let S(6,3) = 2 + 2 + 2

= 3 + 2 + 1

= 4 + 1 + 1

we can see some partition have 1 some partition dosen't.

S(n,k) number of way of partitionig n into k part, in that k part some have 1 some dosen't.

if we take out 1 then we get two cases

1. if it have 1 and take it out we get S(n-1,k-1) ways.
2. if it dosen't have 1 and we take out, it will be S(n-1,k) but we can take out 1 from any k part so we have kS(n-1,k) ways.

so S(n,k) = S(n-1,k-1) + kS(n-1,k)

13. Count the number of 0/1-strings of length n that contains at least (n - 3) consecutive 1s.

Answer:

n length string and atleast (n-3) consecutive 1's, we merge (n-3) consecutive 1's to single 1. Now analyse cases

1. we have 0/1 possibility in first 3 boxes and merged one is in last box, so $2^3 = 8$

0/1	0/1	0/1	1//

2. we slide left to merged one in 3rd box and first two box have 0/1 possibilities but last box have only 0 ,if we consider 1 then it got overlapped with case 1 result so we ignore that case. So $2^2 = 4$ ways

0/1	0/1	1//	0

3. we slide left to merged one in 2rd box so first box have 0/1 possibilities and last two box have 0 0 and 0 1 case if we consider 1 0 and 1 1 then it got overlapped with case 2 and 1 respectivelly, So $2 + 2 = 4$ ways

0/1	1//	0	0
0/1	1//	0	1

4.we slide left to merged one to 1st box and last three box have only 000, 001, 010, 011 case if we take 100, 101, 110, 111 it got overlapped with previous cases so 4 ways.

1//	0	0	0
1//	0	0	1
1//	0	1	0
1//	0	1	1

total ways $= 8 + 4 + 4 + 4 = 20$

15. Cofficient of 1/x and $1/x^2$ in $(x + x^{-1})^n$.
Answer:
$(x + x^{-1})^n = \binom{n}{0}(x^{-1})^n + \binom{n}{1}x(x^{-1})^{n-1} + \binom{n}{2}x^2(x^{-1})^{n-2} + \binom{n}{3}x^3(x^{-1})^{n-3} + \ldots\ldots\ldots\ldots +$ $\binom{n}{k}x^k(x^{-1})^{n-k} + \ldots.\binom{n}{n}x^n$

cofficient of 1/x = $\binom{n}{k}$ when k-n+k = -1.
k = (n-1)/2.
$=\binom{n}{\frac{n-1}{2}}$

cofficient of $1/x^2 = \binom{n}{k}$ when k-n+k = -2.
k = (n-2)/2.
$=\binom{n}{\frac{n-2}{2}}$

16. How many functions are there from $\{1, 2, \ldots\ldots n\}$ to $\{1, 2, \ldots\ldots m\}$?

Answer:
For each element of domain (=n) we can have m different functions.
D = $\{x_1, x_2 \ldots\ldots x_n\}$
R = $\{y_1, y_2 \ldots\ldots y_n\}$
The set of functions S = $\{f(x_1), f(x_2) \ldots\ldots f(x_n)\}$
Now, $f(x_1)$ can be chosen in m ways, $f(x_2)$ can be chosen in m ways, and so on..
The number of functions =
$m * m * m * ... * m$[n times]=m^n

18. Closed form of $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} +$ ....
Answer:
$(1+x)^n = \binom{n}{0}x + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 +$ .... $+ \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$ –eq(1)
put x = 1 in eq(1)
$(2)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} +$ .... $+ \binom{n}{n-1} + \binom{n}{n}$ eq(2)

put x = -1 in eq(1)
$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} +$ .... $+ (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n}$ –eq(3)

add eq(2) and eq(3)

$(2)^n = 2\binom{n}{0} + 2\binom{n}{2} + 2\binom{n}{4}$............

$(2)^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}$............