Discrete Mathematics 10/11/2018

Assignment 5 Solution(19-27)

Instructor: Sourav Chakroborty Scribe : Arnab Saha

19. Count the number of 20 digit integers in which no two consecutive digits are same.
Ans.
In a 20 digit integer the first digit (from the left) cant be 0. So we can choose the first digit from the rest of 9 digits (i.e. 1,2,3,.....,9) in 9 ways. Now for the second digit the first digit cant be chosen, So for the second digit it can be any of the digits between 0 to 9 except the first digit. So the second digit can be chosen in 9 ways too. And for the rest of the places (i.e. from 3rd digit to 20th digit) it can be chosen using the same method of choosing the second digit. So, the total number of 20 digit integers where no two consecutive digits are same is 9*9*9*.........*9 [upto 20th place] = 920 ways.

20.If A and B are two sets such that $|A| = n$ and $|B| = k$ (B is a subset of A) then how many distinct subsets of A are there whose intersection with B has size 1?
Ans.
Now set A has 'n' elements and B has 'k' elements. So the set $\{A - B\}$ contains $(n - k)$ elements.
So the number of distinct subsets of $\{A - B\} = 2^{n-k}$.
let, S be a subset of A such that $|S \cap B| = 1$.
Now, we can choose 1 element from k elements in k ways.
So the total number of subsets $= k * 2^{n-k}$
There also can be a case where the set S contains atmost 1 element(any element from B). So the total number of subsets containing atmost 1 element and $|S \cap B| = 1$ is = k.
So, total number of distinct subsets of A such that intersection with B is size 1 $= k * 2^{n-k} + k$.

21.20 people are sitting around a table. How many ways can we choose 3 person such that no two of the chosen ones are neighbors?
Ans.
Choosing 3 person from 20 people such that no two of the chosen ones are neighbors = Choosing 3 person from 20 people (Choosing 3 people sitting in three consecutive seats + Choose 3 people where 2 are sitting together with the third sitting apart from them)

- Now, there are $\binom{20}{3}$ ways to choose 3 person from 20 people.
- There are 20 ways to choose 3 people sitting in three consecutive seats in a round table.
- There are 20 ways to to choose 2 people sitting in three consecutive seats in a round table. Now to choose the 3rd person we have to exclude 2 + 2 = 4 seats(2 seats for the two person already choosen before and 2 seats for there neighbours). So, there are 20*16 ways to choose 3 people where 2 are sitting together with the third sitting apart from them.

So, choosing 3 person from 20 people such that no two of the chosen ones are neighbors,
$=\binom{20}{3}$ - 20 - (20*16) = $\binom{20}{3}$ - 340 = 800 ways.

22. Prove that $1 + \binom{n}{2} + \binom{n}{4} + . . . + \binom{n}{n} = 3n$. Give a combinational proof.
Ans.
From Binomial Theorem we know,
$(a + x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + ... + \binom{n}{n}x^n$.
Now putting a = 1, x = 2, we get,
$(1 + 2)^n = 1^n + \binom{n}{1}1^{n-1}2 + \binom{n}{2}1^{n-2}2^2 + ... + \binom{n}{n}2^n$
or, $3n = 1 + \binom{n}{1}2 + \binom{n}{2}4 + ... + \binom{n}{n}2n$.[Proved]

23. Give a combinatorial proof of the following:

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$$

Ans.
Let us consider two Binomial expansion,

$(1+x)^a = 1^a + \binom{a}{1}1^{n-1}x + \binom{a}{2}1^{n-2}x^2 + ... + \binom{a}{n}x^n + ... + \binom{a}{a}x^a$ ....(i)

$(1+x)^b = 1^b + \binom{b}{1}1^{n-1}x + \binom{b}{2}1^{n-2}x^2 + ... + \binom{b}{n}x^n + ... + \binom{b}{b}x^b$ ....(ii)

Now, Multiplying (i) and (ii) we get,
$(1+x)^{a+b} = (1^a+\binom{a}{1}1^{n-1}x+\binom{a}{2}1^{n-2}x^2+.....+\binom{a}{n}x^n+.....+\binom{a}{a}x^a)*(1^b+\binom{b}{1}1^{n-1}x+\binom{b}{2}1^{n-2}x^2+.....+\binom{b}{n}x^n+.....+\binom{b}{b}x^b)$

Now, lets find the co efficient of x^n from both side of the equation.
From L.H.S its clear the co efficient is $\binom{a+b}{n}$
From R.H.S the co-eff of x^n is,

$= [1 * \binom{b}{n}] + [\binom{a}{1} * \binom{b}{n-1}] + [\binom{a}{2} * \binom{b}{n-2}] + ... + [\binom{a}{n-1} * \binom{b}{1}] + [\binom{a}{n} * 1]$

$= \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}$

Now, as co efficient of x^n is equal in both sides, we can write,

$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$[proved]

24. Let X = $\{x1, ..., xn\}$ and Y = $\{y1, ..., yk\}$. Count the number of surjections (onto maps) from X to Y.
Ans.
Let e_{nk} denotes the number of surjections from X $\rightarrow$ Y.
The number of functions from X to Y whose ranges exclude atleast m elements in Y is
$b_m = \binom{k}{m}(k-m)^n$
because, m elements out of k elements in set Y can be choosen in $\binom{k}{m}$ ways. And the number of function from X to the remaining $(k-m)$ set is $(k-m)^n$.
We required the number of e_{nk} those function X $\rightarrow$ Y whose ranges excludes precisely none of the elements of Y. By counting backwards from m = k to m = 0, this number is
$e_{nk} = b_0 - (.... - b_{m-2} - (b_{m-1} - b_m))$

$= \sum_{m=0}^{k}(-1)^m b_m$

$= \sum_{m=0}^{k}(-1)^m \binom{k}{m}(k-m)^n$

now replacing m by $(k-m)$ and using $\binom{k}{k-m} = \binom{k}{m}$ in the above expression we get,

$e_{nk} = \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^n$.

So the number of surjections from X $\rightarrow$ Y $= \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^n$

26.Let G be a simple path of length n. A valid coloring of the path is an assignment of colors to the vertices such that no edge is monochromatic (ie. has both end points of the same color). The goal is to compute the number of ways to color the path with five colors (red, green,blue, yellow, violet) in three different scenarios:
(a) There are no more restrictions.

Ans. (a).
As G is a simple path of length n so, the number of vertices in G is (n+1). We have 5 colors. When there is no restriction we can color the first vertex in 5 ways, Second vertex in 4 ways [As no adjacent vertex has same color].

Third vertex also can be colored in 4 ways and so on upto (n+1)th vertex.
So, the total number of ways to color the path with 5 colors is $= 5 * 4^n$.