

Assignment 6

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Show that every connected graph has at least two vertices u and v such that $G - u$ and $G - v$ are connected.

Solution

Let G be a connected graph. We consider the minimum spanning tree of G as T . Since T has no cycle, hence there exists a longest path in T . Let u and v be the two end points of this path. Clearly $T - u$ and $T - v$ are connected.

G is formed by adding edges to T

$G - u$ and $G - v$ must be connected.

Hence Proved.

Problem 2

Show that for any graph $G = (V, E)$ the vertex set V can be partitioned into two sets V_1 and V_2 such that

$$e(V_1) + e(V_2) \leq |E|/2$$

where $e(V_i)$ means the number of edges in E with both end points in V_i .

Solution

We know that any graph $G=(V,E)$ can be reduced to a bipartite graph by removing at most $|E|/2$ edges. We reduce G to a bipartite graph H by removing $|E|/2$ or fewer edges. Let V_1 and V_2 be respectively the set of vertices in the two parts of H . Here $V_1 \cup V_2 = V$. Also V_1 and V_2 are independent sets. Now if we reintroduce the missing edges in H to form G , every edge we introduce will have both end points either in V_1 or V_2 .

number of edges introduced = $e(V_1) + e(V_2)$

$$= \text{number of edges initially removed} \leq |E|/2$$

$$e(V_1) + e(V_2) \leq |E|/2$$

V_1 and V_2 is our required partition.

Hence Proved

Problem 3

Prove that a regular bipartite graph of degree at least 2 does not contain a bridge.

Solution

Let the given statement be false. Let $G=(V,E)$ be a regular bipartite graph. Let $G' = (V', E')$ be a connected component of G such that $e \in E'$ is a bridge. Let x and y be the end points of e .

Since G is bipartite hence $G' - e$ is also bipartite.

By assumption $G' - e$ has two connected components say H_1 and H_2 which are also bipartite.

Assume WLOG that H_1 has vertex x , and that H_2 has vertex y . Let X_1, Y_1 be the bipartition of H_1 and that $x \in X_1$.

Also, let the vertex degree in G be k . Then in G' , x and y have vertex degree $k-1$ (due to subtraction of e) and all remaining vertices have degree k .

Clearly, in sub-graph H_1 , we have

$$S_1 = \sum_{v \in X_1} d(v) = k(|X_1| - 1) + k - 1 \text{ and } S_2 = \sum_{v \in Y_1} d(v) = k(|Y_1|).$$

Thus for $|X_1|, |Y_1| \in \mathbb{N}$, $S_1 \neq S_2$ unless $k=1$. However, it must be the case that $S_1 = S_2$ in a bipartite graph. Therefore, our initial assumption that e is a bridge was wrong. Therefore, G has no bridge.

Hence Proved

Problem 4

Let G be a graph with minimum degree 2. Show that there exist a connected graph with same degree sequence.

Solution

Let us consider a graph $G=(V,E)$ with minimum degree 2 containing 2 connected components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. We know that a graph with minimum degree 2 contains a cycle. Hence both G_1 and G_2 contain a cycle each.

Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be edges that lie on cycles such that $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$

Since e_1 and e_2 lie on cycles, therefore $G_1 - e_1$ and $G_2 - e_2$ are connected. Hence if we remove e_1 and e_2 and introduce edges (u_1, u_2) and (v_1, v_2) we will end up with a single connected graph with the same degree sequence as in G as the degrees of u_1, u_2, v_1, v_2 all first got decremented by 1 and then got incremented by 1. Thus we can merge two connected components of minimum degree 2 to form a connected graph of the same degree sequence. Similarly a graph with minimum degree 2 and any number of connected components can be successively merged into a single connected graph with same degree sequence.

Therefore for a graph with minimum degree 2 there exists a connected graph with same degree sequence.

Hence Proved

Problem 4

Let T_1, \dots, T_k be sub-trees of a tree T such that for all i, j the trees T_i and T_j have a vertex in common. Show that T has a vertex that is in all T_i .

Solution

Let a be the common vertex of T_1 and T_2 .
Let us consider the tree T_i where $i \in \{3, \dots, k\}$.
Let b be the common vertex of T_1 and T_i and let c be the common vertex of T_2 and T_i .
Now there is a path from a to b in $T_1 (a, b \in T_1)$, a path from b to c in $T_i (b, c \in T_i)$ and a path from c to a in $T_2 (c, a \in T_2)$.
Therefore if a, b, c were distinct then they would lie on a cycle in the super graph T which is impossible since T is a tree. Hence a, b, c must be the same. Therefore $\forall i \in \{3, \dots, k\}$ T_i has the same vertex a in common with T_1 and T_2 .
Hence T has a vertex that is in all T_i .

Hence Proved

Problem 6

If $G = (V, E)$ is a graph on n vertices such that all the vertices have even degree. Show that the edge set E can be partitioned into pairwise disjoint sets C_1, C_2, \dots, C_k such that for all $1 \leq i \leq k$ the subgraphs (V, C_i) is a cycle and a collection of isolated vertices.

Hence Proved

Solution

Hint: See Mid Semester solution problem no 13.

Problem 7

Prove that either G or \bar{G} is connected.

Solution

Let $G = (V, E)$.
Now $\bar{G} = (V, \bar{E})$, where \bar{E} = set of all edges which does not belong to E .
If G is connected then we are done. If not to show that \bar{G} is connected.
Let G be not connected.
Let u, v be any two arbitrary points in V . To show that there exists a path between u and v in \bar{G} .

Since, G is disconnected, G has more than one connected component. There are two possibilities

1. u and v belong to different components
2. they both belong to same component.

Case1 : Since u and v belong to different component, there is no edge between u and v in G .

$\implies (u,v)$ belongs to \bar{E}

Case2 : Let w be any vertex that belongs to a component not containing than u and v . Since, u and w belong to different components, there is no edge between u and w .

$\implies (u,w) \in \bar{E}$

Similarly, $(w,v) \in \bar{E}$

\implies There exists a path from u to v in \bar{E}

In either case, there exists a path from u to v in \bar{E}

Since, u and v are arbitrary $\implies \bar{G}$ is connected.

Therefore, either G is connected or \bar{G} is connected.

Hence Proved

Problem 8

A chord of a cycle is an edge that connects two non-adjacent vertices in the cycle. Prove that if every node of G has degree ≥ 3 then G contains a cycle with a chord.

Solution

Proof by Induction on the number of vertices.

Let n be the number of vertices in the graph.

Base Case: $n = 4$

In this case the only possibility is K_4 and it's clear that it has a chord.

Induction Hypothesis: Let the statement is true for any graph with $n-1$ vertices, where $n \geq 5$. To prove that this true for any graph with n vertices.

Induction Process: Let $G=(V,E)$, with n vertices .

Let $u,v \in V$, then $\deg(u), \deg(v) \geq 3$.

Now we merge the two vertices u and v , and name it w , with the condition that if they both have a vertex as a common neighbour then we will delete one edge to avoid multiple edges between two vertices, to make the resultant graph simple and all other edges will remain.

$\implies \deg(w) \geq 5$ and all vertices degree ≥ 3 .

Let the resultant graph will be G_1 .

the number of vertices of G_1 is $n-1$ and every node has degree ≥ 3 .

So, by the induction hypothesis we can say that this graph G_1 contains a cycle C with a chord.

Now, two cases are possible.

1. C contains w
2. C does not contain w

Case 1: Let the $C=(v_1, v_2, \dots, v_{k-1}, w, v_{k+1}, \dots, v_m)$, for some k and m .

if $v_{k-1}, v_{k+1} \in \text{nbd}(u)$, then

$$C_1 = (v_1, v_2, \dots, v_{k-1}, u, v_{k+1}, \dots, v_m).$$

if $v_{k-1}, v_{k+1} \in \text{nbd}(v)$, then

$$C_1 = (v_1, v_2, \dots, v_{k-1}, v, v_{k+1}, \dots, v_m).$$

Let above two conditions not satisfied.

Suppose that $v_{k-1} \in \text{nbd}(u)$ and $v_{k+1} \in \text{nbd}(v)$.

$$\text{Then } C_1 = (v_1, v_2, \dots, v_{k-1}, u, v, v_{k+1}, \dots, v_m).$$

Here C_1 is the cycle of G with a chord.

Case 2: Then C will be the cycle of G containing a chord.

Hence Proved

Problem 9

Prove that there is a tournament T with n players and at least $n!2^{-(n-1)}$ Hamiltonian paths.

Solution

Let $G=(V,E)$, where $V = \{v_1, v_2, \dots, v_n\}$.

Now, $v_{i_1}v_{i_2}\dots v_{i_n}$ is said to be Hamiltonian path if, $v_{i_1} \rightarrow v_{i_2} \rightarrow \dots \rightarrow v_{i_n}$, where $i_j : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ is a bijection.

Now, we can arrange v_1, v_2, \dots, v_n in $n!$ ways and any arrangement will be a Hamiltonian path its probability is $\frac{1}{2^{n-1}}$.

Because, there are $n-1$ edges between n vertices of a particular arrangement, and its direction can be either \rightarrow or \leftarrow and we need only \rightarrow .

i.e. Probability that an edge will have the direction $\rightarrow = \frac{1}{2}$.

i.e. Probability that an arrangement will be Hamiltonian path $= \frac{1}{2^{n-1}}$.

Since, total number of arrangements of vertices is $n!$ then the expected number of Hamiltonian path is equals to $n!2^{-(n-1)}$.

Since, maximum number of Hamiltonian path \geq expected number of Hamiltonian path.

$\implies \forall n, \exists$ a tournament T with at least $n!2^{-(n-1)}$ Hamiltonian path.

Hence Proved