Discrete Mathematics	DATE: 29.07.2018

Solutions of Assignment 1:

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Problem 1: **Let R, S and T be three sets. Answer whether the following statements are true or false. In either case present a proof:**

(a) $(R \cup S) = (R \cup T) \implies (S = T)$

Answer: FALSE

Proof. (Proof by counter example.)
Let, $R = \{1, 2, 3, 4\}$, $S = \{3\}$, $T = \{4\}$.
$R \cup S = \{1, 2, 3, 4\}$
$R \cup T = \{1, 2, 3, 4\}$
$\therefore R \cup S = R \cup T$
But $S \neq T$. □

(b) $(R \subseteq S) \implies ((R \cap T) \subseteq (S \cap T))$

Answer: TRUE

Proof. Let $x \in R$.
As $R \subseteq S$, $x \in S$ too.
If $x \in R \cap T$,
$x \in R$ and $x \in T$
$\implies x \in S$ and $x \in T$
$\implies x \in (S \cap T)$
$\therefore (R \cap T) \subseteq (S \cap T)$
$\therefore (R \subseteq S) \implies ((R \cap T) \subseteq (S \cap T))$. □

(c) $(S \subseteq T) \iff ((S \cap T) = S)$

Answer: TRUE

Proof. This has two parts.

Part 1: To show $(S \subseteq T) \implies ((S \cap T) = S)$
Firstly, it is obvious that always $(S \cap T) \subseteq S$.
So all we have to show is that $(S \subseteq T) \implies (S \subseteq (S \cap T))$.
Let, $x \in S$
$\therefore x \in T$ [as $S \subseteq T$]
So, $x \in S$ and $x \in T$
$\therefore x \in S \cap T$
$\therefore S \subseteq (S \cap T)$

$\therefore (S \cap T) = S$

Part 2: To show $((S \cap T) = S) \implies (S \subseteq T)$
Let, $x \in S$

$\implies x \in (S \cap T)$ [as $S \cap T = S$]
$\implies x \in S$ and $x \in T$
$\therefore x \in T$
$\therefore S \subseteq T$

$\therefore ((S \cap T) = S) \implies (S \subseteq T)$

So $(S \subseteq T) \iff ((S \cap T) = S)$ □

(d) $(R \cup S) = (R \cup T) \iff ((S\text{-}R) = (T\text{-}R))$

Answer: TRUE
Proof. This has two parts.

Part 1: To show $(R \cup S) = (R \cup T) \implies ((S\text{-}R) = (T\text{-}R))$.
Let, $x \in R \cup S$ and $x \notin R$.
$\therefore x \in S - R$
Now, $x \in (R \cup T)$
$\therefore x \in (T - R)$

$\therefore (S - R) \subseteq (T - R)$

Similarly, Let, $x \in R \cup T$ and $x \notin R$.
$\therefore x \in T - R$
Now, $x \in (R \cup S)$
$\therefore x \in (S - R)$

$\therefore (T - R) \subseteq (S - R)$

$\therefore (T - R) = (S - R)$

$\therefore (R \cup S) = (R \cup T) \implies ((S\text{-}R) = (T\text{-}R))$
Now we have to proof the converse.

Part 2: To show $((S\text{-}R) = (T\text{-}R)) \implies (R \cup S) = (R \cup T)$.
$(R \cup S) = R \cup (S - R)$ and similarly, $(R \cup T) = R \cup (T - R)$.
So, if $(S - R) = (T - R)$ we have $(R \cup S) = (R \cup T)$.

□

(e) $P(S \cap T) = P(S) \cap P(T)$
Answer: TRUE

Proof. Let, $x \in S \cap T$
$\therefore x \in P(S \cap T)$

Now, $x \subseteq S \cap T$
$\implies X \subseteq S \wedge x \subseteq T$
$\implies x \in P(S) \wedge x \in P(T)$
$\implies x \in P(S) \cap P(T)$
$\therefore P(S \cap T) \subseteq P(S) \cap P(T)$

Now, let assume,
$x \in P(S) \cap P(T)$
$\therefore x \in P(S) \wedge x \in P(T)$
$\implies x \subseteq S \wedge x \subseteq T$
$\implies x \subseteq S \cap T$
$\implies x \in P(S \cap T)$

$\therefore P(S) \cap P(T) \subseteq P(S \cap T)$

$\therefore P(S \cap T) = P(S) \cap P(T)$. □

(f) $(S \times T)^c = S^c \times T^c$

Answer: FALSE

Proof. Proof by counter example.
Let, $S = \{1, 2\}$, $T = \{3, 4\}$, $U = \{1, 2, 3, 4, 5\}$
$S \times T = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
$|(S \times T)^c| = 21$ [as total 25 pairs are possible]
$\overline{S} = \{3, 4, 5\}, \overline{T} = \{1, 2, 5\}$
$|\overline{S}| \times |\overline{T}| = 3X3 = 9$
$9 \neq 21$. □

Problem 2: **Prove or Disprove:**

(a) The Cartesian product is associative.
Answer: FALSE
Proof. Proof by counter example. Let, $A = \{1\}$
$((A \times A) \times A) = ((1, 1), 1)$.
$(A \times (A \times A)) = ((1, (1, 1))$.
But they are not equal. □

(b) The Cartesian product is commutative.
Answer: FALSE
Proof. Proof by counter example. Let, A={1,2}, B={3,4}
$\therefore A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
and $B \times A = \{(3, 1), (4, 1), (3, 2), (4, 2)\}$
$\therefore A \times B \neq B \times A$. □

(c) The relation “is connected to” for a pair of vertices in an undirected graph is an equivalence relation.
Answer: TRUE

Proof.

- Reflexive: Let V is the set of vertices, $v \in V$.
  Now, v is connected to v.
  $\therefore$ vRv exists.
- Symmetric: Let $u, v \in V$.
  Now, if uRv (u is connected to v) then vRu (v is connected to u) by going through the edges in the reverse order.
- Transitive: Let $u, v, w \in V$.
  Now, if uRv and vRw, then uRw, by first going from u to v and then going from v to w,

  $\therefore$ It is an equivalence relation.

□

(d) The set $\mathbb{Q}^+$ is countable.
Answer: TRUE

Proof. To show that $\mathbb{Q}^+$ is countable we need to show a 1-1 mapping from $\mathbb{Q}^+$ to $\mathbb{N}$.
If $\frac{p}{q} \in \mathbb{Q}^+$, where $p, q \in mathbbN$ and $q \neq 0$ then let us define the bijection as follows.

$$F(\frac{p}{q}) = 2^p(2q-1)$$

All we need to prove is that the function F is 1-1.
Let us prove by contradiction. Let $F(p/q) = F(r/s)$.
Then $2^p(2q-1) = 2^r(2s-1)$.
Since $q, s \geq 1$ so $(2q-1)$ and $(2s-1)$ are both non-zero.
The maximum power of 2 that divides LHS is 2^p while the maximum power of 2 that divided RHS is 2^r.
So if $2^p(2q-1) = 2^r(2s-1)$ then p must be equal to r.
And in that case we have $(2q-1) = (2s-1)$ which means $q = s$.
So if $F(p/q) = F(r/s)$ then $p = r$ and $q = s$.
So F is a 1-1 function and hence $\mathbb{Q}^+$ is countable. □

Problem 3: **Is the relation “is at least as popular as” a valid ordering and if so is it a partial ordering or a total ordering?**
Proof.

Proof of non-transitive: (by counter example)
Let us assume it is transitive.
Therefore if there are 3 contestants A, B and C. Then if $A > B$ and $B > C$ then it

must be $A > C$.
Let there are 3 voters, namely 1, 2 and 3. Their preferences for contestants are as follows in Table.

1	2	3
a	b	c
b	c	a
c	a	b

Table 1: Preference

Now from column 1 and 3, $a > b$, from column 1 and 2, $b > c$.
But from column 2 and 3, $c > a$.
$\therefore$ it is non transitive. So there is no valid ordering. □

Problem 4: **Let R be a relation on S. Prove that $\sum_{x \in S} |N^+(x)| = \sum_{x \in S} |N^-(x)| = m$.**

Proof. Consider the relation as a directed graph.
$\therefore$ if (x,y) $\in R$ then there is an edge $x- > y$.
Suppose the out-degree of x is k. Then these k edges incident on some other nodes.
$\therefore$ out-degree of x adds the same to the total in-degree
$\therefore$ if outdegree of x is k then for entire graph, out degree increases k and indegree increases k too.
$\therefore \sum_{degree} x = indegree = outdegree$
Now $|R| = m$, therefore total m edges. Each edge is connected with some node. Consider vertex x_1, let the edge is m_1, for x_2, let it be m_2, and so on ...
$\therefore m_1 + m_2 + ... = m$ But m_1= Out-degree of x_1, m_2= Out-degree of x_2, and so on.
$\therefore$ sum of total out-degree =m. Similarly for in-degree.
So, $\sum_{x \in S} |N^+(x)| = \sum_{x \in S} |N^-(x)| = m$. □

Problem 5: **How many functions are there from a domain of size n to a range of size m?**

For each element of domain (=n) we can have m different functions.
$D = \{x_1, x_2, ...x_n\}$
$R = \{y_1, y_2, ...y_m\}$
The set of functions S=$\{f(x_1), f(x_2), ..., f(x_n)\}$
Now, $f(x_1)$ can be chosen in m ways, $f(x_2)$ can be chosen in m ways, and so on..
The number of functions =
$m * m * m * ... * m$ [n times]=m^n

Problem 6: **How many one to one functions are there from a domain of size n to a range of size m?**

There can be two cases.

Case 1: If $m \geq n$:
There will be m ways to map 1st element, $m-1$ ways to map 2nd element, $m-2$ ways to map 3rd element, ... , $(m-n+1)$ way to map nth element.
So, $m*(m-1)*(m-2)*....*(m-n+1) = m!/(m-n)! =^m P_n$ number of into functions.

Case 2: If $m < n$:
Not possible.

Problem 7: **How many onto functions are there from a domain of size n to a range of size m?**

Let S be the domain of size n and R be the range of size m, with $m \leq n$. If $m > n$ then the number of onto functions from S to R is 0.
Let $R = \{y_1, \ldots, y_m\}$.
Any function f from S to R is defined by the sets $f^{-1}(y_1), \ldots, f^{-1}(y_m)$.
If f is onto we have the additional property that for all i $f^{-1}(y_i)$ is non-empty.
So the number of onto functions from S is to R is the number of ways the elements of S can be partitioned into m non-empty subsets.

This number is
$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}(m-i)^n.$$

Problem 8: **If f is a function from the set D = {1, 2, 3, . . . , 100} to R = {1, 2, . . . , 200}. How many decreasing functions are there from D to R and how many non-increasing functions are there from D to R.**

If f is an decreasing function then the function is defined by the set $\{f(1), \ldots, f(100)\}$ and given a subset of R of size n there is an unique decreasing function. So the number of decreasing function is $\binom{200}{100}$.

Any function f from D to R is defined by the sets $f^{-1}(1), \ldots, f^{-1}(200)$.
If f is a non-increasing function we have the additional property that for all $i, j \in R$ if $i < j$ then all the elements in $f^{-1}(j)$ is before all the element in $f^{-1}(i)$.
So such a non-increasing function is just defined by the size of $f^{-1}(1), \ldots, f^{-1}(200)$.
So the number of onto functions from S is to R is the number of ways 100 identical objects can be partitioned into 200 subsets. The number is $\binom{299}{100}$.