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Solution of Assignment II

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Problem 1: Write the contrapositive of the following statement:
Given a finite family of convex sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$ (where $n \geq d+1$) such that if the intersection of every $d+1$ of these sets is non-empty, then the whole collection has a nonempty intersection.

Answer: Here , the statement "Given a finite family of convex sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$ (where $n \geq d+1$)" is the set of assumptions(variables) which will hold for the entire problem. So, we can denote this as $\forall x$.

Now consider ,

"if the intersection of every $d+1$ of these sets is non-empty" as $A(x)$ and
"then the whole collection has a nonempty intersection" as $B(x)$.

So, the statement reduces to

$$\forall x\,(A(x) \Rightarrow B(x))$$

Now the contrapositive of this is

$$\forall x\,(\neg B(x) \Rightarrow \neg A(x))\,.$$

So,the contrapositive of the whole statement will be ,

Given a finite family of convex sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$ (where $n \geq d+1$), if the whole collection has an empty intersection then there exist a $d+1$ of these sets which is empty.

Alternative Answer : Consider y : finite family of sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$(where $n \geq d+1$) and $C(y)$: y is convex . So, we can denote "$\forall y C(y)$" as finite family of convex sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$ (where $n \geq d+1$).

Then the statement can be written as

$$\forall y\,(C(y) \Rightarrow (A(y) \Rightarrow B(y)))$$

So, the contrapositive of this can be written as,

$$\forall y\,((\neg\,(A(y) \Rightarrow B(y))) \Rightarrow \neg C(y))$$

$$\forall y\,((A(y) \wedge \neg B(y)) \Rightarrow \neg C(y))$$

Hence, the contrapositive of the whole statement will be,

For a finite family of sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$(where $n \geq d+1$), if the intersection of every $d+1$ of these sets is non-empty and the whole collection has an empty intersection then the family of sets is not convex.

Problem 2: Prove that for any positive reals a, b

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Answer: Given that,

$$a, b \in \mathbb{R}^+$$

Hence , $\sqrt{a}, \sqrt{b}$ exist.

We know that,

$$\left|\sqrt{a} - \sqrt{b}\right| \geq 0$$

$$\left|\sqrt{a} - \sqrt{b}\right|^2 \geq 0$$

$$a + b - 2\sqrt{ab} \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(Proved).

Problem 3: Prove that for any positive reals a, b, c, d

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

Answer: Given that,

$$a, b, c, d \in \mathbb{R}^+$$

Hence$\sqrt{a}$,$\sqrt{b}$,$\sqrt{c}$,$\sqrt{d}$ exist.

We know that,

$$\left|\sqrt{a} - \sqrt{b}\right| \geq 0$$

$$\left|\sqrt{a} - \sqrt{b}\right|^2 \geq 0$$

$$a + b - 2\sqrt{ab} \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Similarly for c and d,

$$\frac{c+d}{2} \geq \sqrt{cd}$$

Now suppose,

$$p = \frac{a+b}{2}$$

$$q = \frac{c+d}{2}$$

As, $a, b, c, d \in \mathbb{R}^+$, Hence , $p, q \in \mathbb{R}^+$

So,

$$\frac{p+q}{2} \geq \sqrt{pq}$$

After putting the value of p and q , We get,

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\frac{(a+b)}{2}\frac{(c+d)}{2}}$$

As,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and

$$\frac{c+d}{2} \geq \sqrt{cd}$$

Hence,

$$\frac{(a+b)}{2}\frac{(c+d)}{2} \geq \sqrt{abcd}$$

So,

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\sqrt{abcd}}$$

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt[4]{abcd}$$

(Proved)

Problem 4: Prove that $\sqrt{2}$ is not rational.

Answer: Let's say ,$\sqrt{2}$ is rational.

$$\sqrt{2} = \frac{p}{q}$$

(Where p and q are integers and $q \neq 0$ and p and q is coprime)
hence,

$$p = \sqrt{2}q$$

$$p^2 = 2q^2 \quad \text{(Squaring both side)}$$

So, p^2 is even $\Rightarrow p$ is even.(As, square of odd number can't be an even number)

Let's say , $p = 2m$

$$\begin{aligned} p^2 &= 2q^2 \\ \Rightarrow (2m)^2 &= 2q^2 \\ \Rightarrow 2m^2 &= q^2 \end{aligned}$$

Hence, q is also even.

As, p and q both are even , they are not coprime(as they have common factor) which is a contradiction.

Hence ,$\sqrt{2}$ is irrational (Proved)

Problem 5: Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

Answer: Let's say ,$\sqrt{2}+\sqrt{3}$ is rational.

Hence ,

$$\begin{aligned} &\sqrt{2}+\sqrt{3} = \frac{p}{q} && \text{(Where p and q are coprime)} \\ &\Rightarrow\sqrt{3} = \frac{p}{q} - \sqrt{2} \\ &\Rightarrow 3 = \left(\frac{p}{q} - \sqrt{2}\right)^2 && \text{(Squaring both side)} \\ &\Rightarrow 3 = \frac{p^2}{q^2} - 2\sqrt{2}\frac{p}{q} + 2 \\ &\Rightarrow\sqrt{2} = \frac{q}{2p}\left(\frac{p^2}{q^2} - 1\right) \\ &\Rightarrow\sqrt{2} = \frac{p}{2q} - \frac{q}{2p} \end{aligned}$$

As , p and q are rational, right side of this expression is rational.

But left side of the expression is $\sqrt{2}$ which is irrational (Proved earlier)
Which can't be true at the same time.

Hence , $(\sqrt{2}+\sqrt{3})$ is irrational.(Proved)

Problem 6: Is the statement $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$ a tautology or a contradiction or none.

Answer:

$$\begin{aligned}
&(p \wedge q) \vee (\neg p \vee (p \wedge \neg q)) & \\
&\Rightarrow (p \wedge q) \vee ((\neg p \vee p) \wedge (\neg p \vee \neg q)) & \text{(Using Distributive Law)} \\
&\Rightarrow (p \wedge q) \vee (T \wedge (\neg p \vee \neg q)) & \text{(Where T is Tautology)} \\
&\Rightarrow (p \wedge q) \vee (\neg p \vee \neg q) & \\
&\Rightarrow (p \wedge q) \vee \neg(p \wedge q) & \text{(Using Demorgan's law)} \\
&\Rightarrow T & \text{(Where T is Tautology)}
\end{aligned}$$

Hence the statement is Tautology.

Problem 7: Prove that there are infinitely many primes of the form $3(mod4)$.

Answer: Let's say there are finite number of primes of the form $3(mod)4$. They are $P_1, P_2, ..., P_k$.

Now consider a number $P = P_1 P_2 ...... P_k$. So P can be of the form $(4k+1)$ or $(4k+3)$.

Case 1 : When P is of the form $(4k+1)$, then consider a number $P' = (P+2)$ (Which is of the form $(4k+3)$)

We know that $\forall i(1 \leq i \leq k)$, $P_i | P$.and if $P_i | P'$ then P_i has to divide 2 but no prime of the form $(4k+1)$ divides 2. (As, 2 is the smallest prime and not of the form $(4k+1)$)
Hence $\forall i(1 \leq i \leq k)$, $P_i \not| P'$.

Case 2 : When P is of the form $(4k+3)$, then consider a number $P" = (P+4)$(Which is also of the form $(4k+3)$)

We know that $\forall i(1 \leq i \leq k)$, $P_i | P$.and if $P_i | P"$ then P_i has to divide 4 but no prime of the form $(4k+1)$ divides 4. (As, 2 is the only prime which divides 4 but not of the form $(4k+1)$)
Hence $\forall i(1 \leq i \leq k)$, $P_i \not| P"$.

So P' and $P"$ (which are of the form $(4k+3)$) doesn't have any prime factor of the form $(4k+3)$.
So , P' and $P"$ can have prime factor only of the form $(4k+1)$.
Taking prime factorization of P' and P", we get,

$$P' = (r_1^{\alpha_1} r_2^{\alpha_2} ... r_k^{\alpha_k})$$

and

$$P" = (m_1^{\beta_1} m_2^{\beta_2} ... m_k^{\beta_k})$$

(where all $r_i's$ and $m_i's$ are of the form (4k+1)).

We know that multiplication of all the numbers of the form $(4k+1)$ is always of the form $(4k+1)$.
As P' and $P"$ are of the form $(4k+3)$, Atleast one of the $r_i's$ and one of the $m_i's$ has to be of the form $(4k+3)$ but we have proved that no prime of the form $(4k+3)$ divides P' and $P"$ which is a contradiction.

Hence , Number of primes of the form $3(mod)4$ are infinite.(Proved)

Problem 9: Write the opposite of the following statement: There is an university in USA where every department that has at least 20 faculty has at least one noble laureate.

Answer: This statement can be written as,

$$\exists\ university\ \forall departments\,(number\ of\ faculties \geq 20) \Rightarrow (atleast\ one\ nobel\ laureate)$$

So,the opposite of the statement is,

$$\neg\,(\exists\ university\ \forall departments\,(number\ of\ faculties \geq 20) \Rightarrow (atleast\ one\ nobel\ laureate))$$

$$\Rightarrow\ \forall\, universities\,(\ \neg\,(\forall departments\,(number\ of\ faculties \geq 20) \Rightarrow (atleast\ one\ nobel\ laureate)))$$

$$\Rightarrow\ \forall\, universities\,\exists\, department\,(\neg\,((number\ of\ faculties \geq 20) \Rightarrow (atleast\ one\ nobel\ laureate)))$$

$$\Rightarrow\ \forall\, universities\,\exists\, department\,((number\ of\ faculties \geq 20)\ \wedge\ \neg\,(atleast\ one\ nobel\ laureate))$$

$$\Rightarrow\ \forall\ universities\ \exists\ department\,((number\ of\ faculties \geq 20)\ \wedge\ \neg\,(no\ nobel\ laureate))$$

Hence,the answer is :

$$"for\ all\ universities\ in\ USA\ there\ exist\ a\ department\ that\ has\ atleast\ 20\ faculty\ has\ no\ nobel\ laureate"$$

Problem 10: What is the contrapositive of the statement For all $C, D, E, F \geq 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have, $C2^n > Dn^8$ and $E(logn)^4 < F(n^{\frac{1}{100}})$

Answer: This statement can be written as,

$$\forall C, D, E, F \geq 0 \Rightarrow \exists N, \forall n > N\ \left(\left(C2^n > Dn^8\right)\ and\ \left(E(logn)^4 < F(n^{\frac{1}{100}})\right)\right)$$

The contrapositive of this is,

$$\neg\left(\exists N, \forall n > N\left(\left(C2^n > Dn^8\ \right) and\ \left(E(logn)^4 < F(n^{\frac{1}{100}})\right)\right)\right) \Rightarrow \neg\forall\,(C, D, E, F \geq 0)$$

$$\Rightarrow \left(\forall N, \exists n > N \left(\neg \left((C2^n > Dn^8\ )\ and\ \left(E(logn)^4 < F(n^{\frac{1}{100}})\right)\right)\right)\right) \Rightarrow \neg\forall\,(C, D, E, F \geq 0)$$

$$\Rightarrow \left(\forall N, \exists n > N \left((C2^n \leq Dn^8\ )\ or\ \left(E(logn)^4 \geq F(n^{\frac{1}{100}})\right)\right)\right) \Rightarrow ((\exists\ C \vee \exists\ D \vee \exists\ E \vee \exists\ F) < 0)$$

Hence the answer is :

For all N , there exist an $n \geq N$ such that $\left((C2^n \leq Dn^8\ )\ or\ \left(E(logn)^4 \geq F(n^{\frac{1}{100}})\right)\right)$ implies atleast one of $C, D, E, F < 0$