Discrete Mathematics 19/8/2018

Assignment 3

Instructor: Sourav Chakraborty *Scribe: Spandan Das*

Answers to problems 1-7

1. *Proof.* We prove this by induction on n. Let,

$$P(n) := \text{For } n \in \mathbb{N}, \quad (1+x)^n \geqslant 1 + nx \quad \forall x > -1$$

Base: $P(1)$ is true since,

$$1 + x \geqslant 1 + x \quad \forall x > -1$$

Assuming $P(n)$ to be true we proceed to prove $P(n+1)$.

$$\begin{aligned}
&\forall x > -1, \\
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\geqslant (1+x)(1+kx) \quad (\text{from } P(k)) \\
&= 1 + x + kx + kx^2 \\
&= 1 + (k+1)x + kx^2 \\
&\geqslant 1 + (k+1)x \quad (\text{since } x^2 > 1)
\end{aligned}$$

$\square$

2. *Proof.* We prove this by induction on n. Let,

$$P(n) := \text{For } n \in \mathbb{N}, \quad \sum_{i=1}^{n} i.i! = (n+1)! - 1$$

Base: $P(1)$ is true since,

$$1.1! = 1 = 2! - 1$$

Assuming $P(n)$ to be true we proceed to proof $P(n+1)$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i.i! &= \sum_{i=1}^{n} i.i! + (n+1).(n+1)! \\
&= (n+1)! - 1 + (n+1).(n+1)! \\
&= (n+1)!(n+1+1) - 1 \\
&= (n+2)(n+1)! - 1 \\
&= (n+2)! - 1
\end{aligned}$$

$\square$

3. *Proof.* We prove by induction on n. Let,

$$P(n) := \text{Given } a_1 = 5,\ a_2 = 13 \text{ and } a_{k+2} = 5a_{k+1} - 6a_k\ \forall k \in \mathbb{N}, \quad a_n = 2^n + 3^n$$

Assuming $P(m)$ to be true we proceed to prove $P(m+1)$.

$$\begin{aligned} a_{m+1} &= 5a_m + 6a_{m-1} \quad \text{(by assumption)} \\ &= 5(2^m + 3^m) - 6(2^{m-1} + 3^{m-1}) \quad \text{(by P(m))} \\ &= (5.2^m - 6.2^{m-1}) + (5.3^m - 6.3^{m-1}) \\ &= (5-3)2^m + (5-2)3^m \\ &= 2.2^m + 3.3^m \\ &= 2^{m+1} + 3^{m+1} \end{aligned}$$

□

4. Let us observe the problem in terms of graph theory. We define $G(V, E)$ an undirected graph as follows,

$$\begin{aligned} V &= \{v | v \text{ is a person}\} \\ E &= \{(u, v) | u \text{ knows } v\} \end{aligned}$$

Then by definition, $R(p, q)$ is the smallest natural number n such that a graph with n vertices has either a size-p clique or a size-q independent set.

   (a) To prove,

$$R(p+1, q+1) \leqslant R(p, q+1) + R(p+1, q)$$

   Proof. Let $R(p, q+1) + R(p+1, q) = n$. We need to prove that for any graph G on n vertices, G either has a size-$(p+1)$ clique or a size-$(q+1)$ independent set. Let us choose any vertex $v \in V_G$. We define,

$$\begin{aligned} N_v &= \{u \in V_G | (u, v) \in E_G\} \\ \bar{N_v} &= \{u \in V_G | (u, v) \notin E_G\} \end{aligned}$$

   Now either $|N_v| \geqslant R(p, q+1)$ or $|\bar{N_v}| \geqslant R(p+1, q)$. Because if not then

$$\begin{aligned} |V_G| &= |N_v| + |\bar{N_v}| + |\{v\}| \\ &\leqslant R(p, q+1) - 1 + R(p+1, q) - 1 + 1. \\ &< n \end{aligned}$$

   This is a contradiction.

      i. **Case I**: If $|N_v| \geqslant R(p, q+1)$ then the induced subgraph of G on the vertex set N_v has a size-p clique or a size-$(q+1)$ independent set. Thus this subgraph along with vertex v has either a size-$(p+1)$ clique or a size-$(q+1)$ independent set. Hence G will have either a size-$(p+1)$ clique or a size-$(q+1)$ independent set.

ii. **Case II**: If $|\bar{N}_v| \geqslant R(p+1,q)$ then the induced subgraph of G on the vertex set $\bar{N}_v$ has a size-$(p+1)$ clique or a size-q independent set. This subgraph along with vertex v has either a size-$(p+1)$ clique or a size-$(q+1)$ independent set. Hence G will have either a size-$(p+1)$ clique or a size-$(q+1)$ independent set.

Hence we have proved by exclusive case studies that

$$R(p+1,q+1) \leqslant R(p,q+1) + R(p+1,q).$$

□

(b) To prove,

$$R(p,q) \leqslant C_{p-1}^{p+q-2} \quad \forall p,q \in \mathbb{N}$$

Proof. We prove this by induction on $(p \wedge q)$. Let,

$$\begin{aligned} P(i) &:= R(i,q) \leqslant C_{i-1}^{i+q-2} \quad \forall q \in \mathbb{N} \\ Q(i) &:= R(p,i) \leqslant C_{p-1}^{p+i-2} \quad \forall p \in \mathbb{N} \end{aligned}$$

Base: $(P(1) \wedge Q(1))$ is true since,

$$\begin{aligned} R(1,q) &= 1 \quad \forall q \in \mathbb{N} \quad \text{(by assumption)} \\ R(p,1) &= 1 \quad \forall p \in \mathbb{N} \quad \text{(by assumption)} \end{aligned}$$

Assuming $(P(k) \wedge Q(l))$ to be true, we proceed to prove $(P(k+1) \wedge Q(l+1))$.

$$\begin{aligned} R(k+1,l+1) &\leqslant R(k+1,l) + R(k,l+1) \quad \text{(from 4.a)} \\ &\leqslant C_k^{k+l-1} + C_{k-1}^{k+1-l} \quad [\text{by } (P(k) \wedge Q(l))] \\ &= C_k^{k+l} \quad [\text{since } C_r^n + C_{r-1}^n = C_r^{n+1}] \end{aligned}$$

Hence by induction we are done. □

5. *Proof.* Let $G(V,E)$ be an undirected graph, where,

$$\begin{aligned} V &= \{v_1, \ldots, v_{2n} \mid v_i \text{ is a participant for each } i = 1, \ldots, 2n\} \\ E &= \{v_i \sim v_j \mid \text{if } v_i \text{ has shaken hand with } v_j\} \end{aligned}$$

Then we need to prove that if G is triangle free graph then $\mid E \mid \leqslant n^2$.
We prove this by the induction on n.

$$\underline{\text{Base}} : n = 2, \text{ then there can be at most 1 handshake.}$$
$$\Rightarrow \mid E \mid \leqslant 1 \leqslant 2^2. \text{ Hence the claim is true for base case.}$$

Assuming the claim to be true for $n = 2k$, let us now take any undirected graph $G'(V_{G'}, E_{G'})$ on $2(k+1)$ vertices. By assumption the graph is triangle free.
Let $u \sim v$ be any edge in G'. If there does not exist any such u and v, then $\mid E_{G'} \mid = 0$ and our claim is trivially true.

Now $G' - \{u, v\}$ is a graph on $2k$ vertices and by induction hypothesis has atmost k^2 edges. Since $G' - \{u, v\}$ is a triangle free as well. Now since $u \sim v$, if $\exists \quad w \in V_{G'}$ such that $w \sim u$ and $w \sim v$, then G' would have a triangle which is not possible. Thus u and v cannot have a common neighbor.

$\Rightarrow$

$$\begin{aligned} N_u &= \{w \in V_{G'} \mid u \sim w\} \text{ and} \\ N_v &= \{w \in V_{G'} \mid v \sim w\} \text{ has no intersection} \end{aligned}$$

$\Rightarrow$

$$\mid N_u \mid + \mid N_v \mid \leqslant 2k \text{ (since } G' - \{u, v\} \text{ has } 2k \text{ vertices)}$$

Thus,

$$\begin{aligned} \mid E_{G'} \mid \geqslant &\mid E_{G'-\{u,v\}} \mid + 2k + 1 \quad \text{(1 is added since } u \sim v\text{)} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

□

6. (a) *Proof.* We prove by induction on m.

Base: $m = 1$

Then only choice for r is 0.

$$\begin{aligned} x_{0+1}x_{1-0} + x_0 x_{1-0-1} &= x_1^2 + x_0 x \\ &= 1 \\ &= x_1 \end{aligned}$$

The claim is true for base case. Assuming the claim to be true for all $m \leqslant k$, let us now observe what happens for $m = k + 1$.

$$x_{k+1} = x_k + x_{k-1} \text{ (by definition of Fibonacci sequence)}$$

Case 1: If $r \leqslant k - 2$, by induction hypothesis we can write,

$$x_k = x_{r+1}x_{k-r} + x_r x_{k-r-1}$$

$$x_{k-1} = x_{r+1}x_{k-1-r} + x_r x_{k-r-2}$$

Thus,

$$\begin{aligned} x_{k+1} = x_k + x_{k-1} &= x_{r+1}(x_{k-r} + x_{k-r-1}) + x_r(x_{k-r-1} + x_{k-r-2}) \\ &= x_{r+1}x_{(k+1)-r} + x_r x_{(k+1)-r-1} \end{aligned}$$

Case 2: If $r = k - 1$, then

$$x_{r+1}x_{(k+1)-r} + x_r x_{(k+1)-r-1} = x_k x_2 + x_{k-1}x_1 = x_{k+1}$$

Case 3: If $r = k$, then

$$x_{r+1}x_{(k+1)-r} + x_r x_{(k+1)-r-1} = x_{k+1}x_1 + x_k x_0 = x_{k+1}$$

Hence the claim holds for $m = k+1$ $\forall r$ such that $0 \leqslant r \leqslant k$. Thus by induction we are done. $\square$

(b) *Proof.* We prove by induction on n. $P(n) := x_d \mid x_{nd}$ for $n \in \mathbb{N}$ and $\forall d \in \mathbb{N}$.
Base: $n = 1$
$x_d | x_d$ trivially holds $\forall d \in \mathbb{N}$. Hence $P(1)$ holds.
Assume that $P(k)$ is true for some $k \geqslant 1$. For any $d \in \mathbb{N}$,

$$d \leqslant (k+1)d - 1 \quad (k \in \mathbb{N}).$$

Using previous question (6.a),

$$x_{(k+1)d} = x_{d+1}x_{kd} + x_d x_{k-d}.$$

By induction hypothesis $x_d \mid x_{kd}$. Hence $x_d \mid x_{(k+1)d}$.
Thus $P(k+1)$ holds provided $P(k)$ holds. Hence by induction we are done. $\square$

7. (a) *Proof.* We prove this by induction on number of vertices of the graph (n).
$P(n)$: Any graph on n vertices with maximum degree k is $(k+1)$ colorable
Base: $n = 1$. Then the graph is 1-colorable. Hence it is $(k+1)$ colorable for any $k \in \mathbb{N} \cup \{0\}$. Thus $P(1)$ is true.
Let us assume $P(m)$ is true. Now let us take any graph G on $(m+1)$ vertices with max degree k. Let v be a vertex in G. Consider the graph $G \backslash \{v\}$. By induction hypothesis $G \backslash \{v\}$ is $(k+1)$ colorable. Now when v is included in $G \backslash \{v\}$ to get G, v has at most k neighbors, those have at most k different color. Thus one color is remaining which can be assigned to v without violating coloring condition. Thus G is $(k+1)$-colorable as well. $P(m) \Rightarrow P(m+1)$, hence we are done. $\square$

(b) *Proof.* We will prove by induction on number of vertices of the graph (n).
$P(n)$:= Any connected graph on n vertices with degree less than or equal to 3 and at least one vertex with degree < 3 is 3 colorable.
Base: $P(1)$ is trivially true since only 1 vertex is there, the graph can be colored by single color.
Assume $P(m)$ is true. Now consider any graph $G = (V, E)$ on $(m+1)$ vertices with degree $\leqslant 3$ and at least one vertex of degree < 3. Let v be a vertex of G which has minimum degree. Since G is connected $\exists u \neq v$ in V such that $u \sim v$. Consider the graph $G \backslash \{u\}$. Clearly this graph will satisfy the condition for $P(m)$ and hence it is three colorable. Let $G \backslash \{u\}$ be colored optimally.
If u has less than two neighbors, then neighbors of u can be colored by at most two colors. Hence the third color can be assigned to u and G remains 3-colorable. Suppose u has 3 neighbors, two neighbors other than v are v_1, v_2, say. In the worst possible case let v_1 and v_2 have different colors C_1 and C_2 respectively. Now since v has at most 1 more neighbor other than u, v cannot be adjacent to

vertices colored both C_1 and C_2 in $G\backslash\{u\}$. Thus for optimal coloring of $G\backslash\{u\}$, it is always possible to color v with either C_1 or C_2. Hence none of the neighbors of u has been assigned the color C_3, which means u can be colored C_3 without violating the coloring strategy. The graph G remains to be 3-colorable.

Hence by induction G is 3-colorable. $\square$