Discrete Mathematics	16/09/2018

Assignment 4 Solution

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2. Prove or Disprove the set of asymptotic relations.

(a) $2.9^{\log_2 n} = \Theta(n^{\log_2 3})$

$\lim_{n\to\infty} \frac{n^{\log_2 3}}{2.9^{\log_2 n}} = \lim_{n\to\infty} \frac{n^{\log_2 3}}{n^{\log_2 2.9}} \sim \lim_{n\to\infty} n^{0.049} \to \infty$

Disproved

(b) $\log\log n = \Omega((\log\log\log n)^{\log\log\log n})$

Let $x = \log\log n$ and $y = \log\log\log n^{\log\log\log n}$

Hence, $\lim_{n\to\infty} \frac{\log y}{\log x} = \lim_{n\to\infty} \log\log\log\log n \to \infty$

Disproved

(c) $n^4 \sim (1-\frac{1}{n})^n n^3$

Let $x = n^4$ and $y = (1-\frac{1}{n})^n n^3$

Then $\log x = 4\log n$ and $\log y = n\log(1-1/n) + 3\log n$

$\lim_{n\to\infty} \frac{\log y}{\log x} = \lim_{n\to\infty} \frac{n\log(1-1/n)}{4\log n} + \frac{3\log n}{4\log n} = \frac{3}{4}$

Hence, they are not asymptotically equal.

(d) $2^{(\log n)-(\log\log n)} \sim 2^{(1-1/n)\log n}$

Let $x = 2^{(\log n)-(\log\log n)}$ and $y = 2^{(1-1/n)\log n}$

Then $\log x = (\log n - \log\log n)\log 2$ and $\log y = (1-1/n)\log n \log 2$

$\lim_{n\to\infty} \frac{\log x}{\log y}$

$= \lim_{n\to\infty} \frac{1/n - 1/(n\log n)}{1/n - 1/n^2 + \log n/n^2}$ [Using L'Hopital's Rule]

$= \lim_{n\to\infty} \frac{1-1/\log n}{1-1/n+\log n/n}$

$= 1$

Hence, they are asymptotically equal.

3. Prove that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$

$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} \sim \frac{\sqrt{2\pi(2n)}(\frac{2n}{e})^{2n}}{(\sqrt{2\pi n}(\frac{n}{e})^n)^2}$ [Using Stirling's Approximation]
Hence, $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$

4. For the following pairs give correct asymptotic relations

(a) $(\log n)^a$ **and** n^b

Claim: $(\log n)^a \leq cn^b$

For the claim to be true we need to find c s.t. $c \geq \frac{(\log n)^a}{n^b}$

Let, $f(x) = \frac{(\log x)^a}{x^b}$

Therefore $f'(x) = \frac{a(\log x)^{a-1}}{x^{b+1}} - \frac{b(\log x)^a}{x^{b+1}}$

and $f''(x) = \frac{a(a-1)(\log x)^{a-2} - a(2b+1)(\log x)^{a-1} + b(b+1)(\log x)^a}{x^{b+2}}$

$f'(x) = 0 \Rightarrow \log x = \frac{a}{b}$, i.e. $x = e^{\frac{a}{b}}$

$f''(e^{\frac{a}{b}}) \leq 0 \ \forall b \in \mathbb{R}$

Therefore $f(x)$ is maximum at $x = e^{\frac{a}{b}}$

If we choose $c = f(e^{\frac{a}{b}})$ then it will satisfy our claim.

Hence $(\log n)^a = O(n^b)$

(b) $2^{n \log_2 n}$ **and** $10n!$

$2^{n \log_2 n} = n^n$ and $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$

Now $\lim_{n\to\infty} \frac{n^n}{10n!} = \lim_{n\to\infty} \frac{e^n}{10\sqrt{2\pi n}} \to \infty$

Therefore, $2^{n \log_2 n} = \omega(10n!)$

(c) $\sqrt{n}$ **and** $(\log_2 n)^5$

$\lim_{n\to\infty} \frac{\sqrt{n}}{(\log_2 n)^5} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{5(\log_2 n)^4}{n \log_e 2}}$ [Using L'Hopital's Rule]
$= \frac{\log_e 2}{10} \lim_{n\to\infty} \frac{\sqrt{n}}{(\log_2 n)^4} = \frac{(\log_e 2)^2}{80} \lim_{n\to\infty} \frac{\sqrt{n}}{(\log_2 n)^3} = ... = \frac{(\log_e 2)^5}{2^5 5!} \lim_{n\to\infty} \sqrt{n} \to \infty$
Therefore, $\sqrt{n} = \omega(\log_2 n)^5$

(d) $n^2/\log_2 n$ **and** $(n\log_2 n)^4$

$\lim_{n\to\infty} \frac{n^2/\log_2 n}{(n\log_2 n)^4} = \lim_{n\to\infty} \frac{1}{n^2(\log_2 n)^5} \to 0$
Therefore, $n^2/\log_2 n = o(n\log_2 n)^4$

(e) $\log_2 n$ **and** $\log_2 66n$

$\lim_{n\to\infty} \frac{\log_2 n}{\log_2 66n} = \lim_{n\to\infty} \frac{\log_2 n}{\log_2 n + \log_2 66} \to 1$
Therefore, $\log_2 n = \Theta(\log_2 66n)$

(g) n^2 **and** $n(\log_2 n)^{15}$

$\lim_{n\to\infty} \frac{n^2}{n(\log_2 n)^{15}} = \lim_{n\to\infty} \frac{n}{(\log_2 n)^{15}} = \frac{\log_e 2}{15} \lim_{n\to\infty} \frac{n}{(\log_2 n)^{14}}$ [Using L'Hopital's Rule]
$= ... = \frac{(\log_e 2)^{15}}{15!} \lim_{n\to\infty} n \to \infty$
Therefore, $n^2 = \omega(n(\log_2 n)^{15})$

5. Find approximate value of $\binom{n}{n/3}$

$\binom{n}{n/3} = \frac{n!}{(n/3)!(2n/3)!} \sim \frac{\sqrt{2\pi n}(n/e)^n}{\sqrt{2\pi(n/3)}(n/3e)^{n/3}\sqrt{2\pi(2n/3)}(2n/3e)^{2n/3}}$ [Using Stirling's Approximation]
$= \frac{3^{n+1}}{2^{2n/3}\sqrt{4\pi n}}$

6. Solve the following recurrences

(a) $T(n) = 2T(\lceil n/2 \rceil) + 5$

Claim: $T(n) \leq cn - d$ for some +ve c and d which will be fixed later

Proof:

From the base case T(2) = 7 condition we have $2c - d \geq 7$

$T(n) = 2T(\lceil n/2 \rceil) + 5$

$\leq 2c\lceil n/2 \rceil - 2d + 5$ [According to I.H.]

$\leq 2cn/2 + c + 5 - 2d$

$= cn - 2d + c + 5$

$\leq cn - d$ [For $c + 5 - d \geq 0$]

$c = 12$ and $d = 17$ satisfies both the condition.

Hence, $T(n) \leq 12n - 17$ $\forall n \geq 2$

(b) $T(n) = T(n-1) + T(n-2) + 15$

Claim: $T(n) \leq c2^n \quad \forall n \geq n_0$ where n_0 will be decided later

Proof:

We have $T(2) = 17$ and $T(3) = 33$. Hence $4c \geq 17$ and $8c \geq 33$ i.e. $c \geq 5$ satisfies both the condition.

$$
\begin{aligned}
T(n) &= T(n-1) + T(n-2) + 15 \\
&\leq c2^{n-1} + c2^{n-2} + 15 \qquad \text{[According to I.H.]} \\
&= c2^n + 15 - c(1 + 1 + 2 + 2^2 + 2^3 + ... + 2^{n-3}) \\
&= c2^n + 15 - c2^{n-2} \\
&\leq c2^n \qquad \text{when } 15 - c2^{n-2} \leq 0
\end{aligned}
$$

Therefore, $c2^{n-2} \geq 15 \Rightarrow 2^{n-2} \geq 3 \Rightarrow n \geq 4$

For $n < 4$ our claim is true as we can check manually and from principle of mathematical induction claim is true for $n \geq 4$ as well

(c) $T(n) = 2T(\lceil n/3 \rceil) + n^2$

Claim: $T(n) \leq cn^2$ for some +ve c which will be decided later

Proof:

We have $T(1) = 1$, $T(2) = 5$, $T(3) = 11$, $T(4) = 26$. $c = 2$ satisfies our assumption for these values of n.

$$
\begin{aligned}
T(n) &= 2T(\lceil n/3 \rceil) + n^2 \\
&\leq 2c\lceil (n/3)^2 \rceil + n^2 \qquad \text{[According to I.H.]} \\
&\leq 2c(n/3 + 1)^2 + n^2 \\
&= 2c(n^2/9 + 2n/3 + 1) + n^2 \\
&= cn^2 - (7cn^2/9 - 4n/3 - 2c - n^2) \\
&\leq cn^2 \qquad \text{when } 7cn^2/9 - 4n/3 - 2c - n^2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&7cn^2/9 - 4n/3 - 2c - n^2 \geq 0 \\
\Rightarrow\ &14n^2/9 - 4n/3 - 4 - n^2 \geq 0 \\
\Rightarrow\ &5n^2 - 12n - 36 \geq 0
\end{aligned}
$$

For $n \geq 5$ the above condition satisfies.

Hence by Principle of Mathematical Induction $T(n) \leq 2n^2 \quad \forall n \geq 1$