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Assignment 5 (First 9 ques.)

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1

Situation is equivalent to dividing n identical balls into k distinct buckets.

Coefficient of x^n in the expansion of $(x + x^2 + x^3 + ...)^k$

$\implies$ Coefficient of x^{n-k} in the expansion of $(1 + x + x^2 + ...)^k$

$\implies$ Coefficient of x^{n-k} in the expansion of $(\frac{1}{1-x})^k$

$\implies$ Coefficient of x^{n-k} in the expansion of $(1 - x)^{-k}$

$\implies {}^{(k)+(n-k)+1}C_{n-k}$

$\implies {}^{n+1}C_{n-k}$

2

Number of shortest path from (0,0) to (n,k) equals the number of permutations possible for n $\rightarrow$ (horizontal arrows) and k $\uparrow$ (vertical arrows)

$\implies \frac{(n+k)!}{n!k!}$

3

$$(A + B)^n = \sum_{k=0}^{n} {}^nC_k A^k B^{n-k}$$

$$\implies (\frac{2}{3} + 1)^n = \sum_{k=0}^{n} {}^nC_k \frac{2}{3}^k 1^{n-k}$$

$$\implies (\frac{2}{3} + 1)^n = \sum_{k=0}^{n} {}^nC_k \frac{2}{3}^k$$

4

4.1 strictly increasing function

- case 1: If n < k

  No functions possible

- case 2: If n $\geq k$

Num of such functions possible = ${}^nC_k$

We plan to pick up k distinct values from the later set. Once we have the k values, the function can be just one permutation of the k values (where all the values are arranged in an increasing order). So, total number of such functions will be number of ways of choosing k objects from a set of n objects.

4.2 increasing function

- case 1: If n < k

  No functions possible

- case 2: If n $\geq$ k

  Num of such functions possible = ${}^{(n+k-1)}C_k$

We have k distinct buckets, which need to be filled with n identical objects (empty buckets allowed). This would give us the number of pre-images for each of the elements in the later set. There can be only one permutation for one such allocation *(with the same logic used in the case of strictly increasing func.)* So, total number of such functions will be number of ways of allocating k objects to n distinct buckets.

5

Let f(n) be the number of reqd. strings of length n.

- case 1 : The string ends in B Number of such instances are f(n-1). As we can always add B at the end of the string. The resultant string would always be a valid string.

- case 2 : The string ends in A the strings must have their second last element as B. num. of such strings must be f(n-2). By logic mentioned in case 1.

  Therefore

$$f(n) = f(n-1) + f(n-2)$$

6

Coefficient of x^n in the expansion of $(x^2 + x^3 + x^4....)^k$
$\implies$ Coefficient of x^{n-2k} in the expansion of $(1 + x + x^2 + ...)^k$
$\implies$ Coefficient of x^{n-2k} in the expansion of $(\frac{1}{1-x})^k$
$\implies$ Coefficient of x^{n-2k} in the expansion of $(1-x)^{-k}$
$\implies$ ${}^{(k)+(n-2k)+1}C_{n-2k}$
$\implies$ ${}^{n-k+1}C_{n-2k}$

7

Num. of ways of partitioning 5 books into 1 bundle = 1

Num. of ways of partitioning 5 books into 2 bundle =

$$\frac{2^5 - 2}{2} = 2^4 - 1$$

Num. of ways of partitioning 5 books into 3 bundles is $S_k(n)$; where n is number of books and k is the number of bundles.

$$S_k(n) = kS_k(n-1) + S_{k-1}(n-1)$$
$$= S_k(5) = 3S_3(4) + S_2(4)$$
$$= S_3(5) = 3S_3(4) + S_2(4)$$
$$= S_2(4) + 3[S_2(3) + 2S_3(3)]$$
$$S_3(5) = 25$$

Total num of ways = 15 + 25 + 1 = 41.

8

First we distribute 3 balls (out of 5 balls) to each of the distinct buckets in $^5C_3 3!$

The remaining 2 balls have 3 options of going to any of the three distinct bucket in 3^2 ways.

Reqd. num of ways are $^5C_3 3! 3^2$

9

Number of paths passing through i vertices (including u,v,w)

= num of ways of dividing k=i-3 distinct balls into 2 distinct buckets (say u-w and w-v) where ordering matters (with empty buckets allowed).

$=^{n-3} C_k k!^{2+k-1}C_k$

$=^{n-3} C_k k!^{k+1}C_k$

As num. of ways of choosing k vertices out of n-3 vertices = $^{n-3}C_k$

Now k varies from 0 to n-3. Therefore total required num of ways

$$= \sum_{k=0}^{n-3} {}^{n-3}C_k k!^{k+1}C_k$$