Midterm
MTech CS Discrete Mathematics, 2018

September, 2018

Time: 3 hours, Maximum Marks: 100

Answer as many questions as possible but the maximum possible marks one can obtain is 100. Your answers should be well-written and you should explain your arguments properly.

1. (5 marks) Let $x \in \mathbb{R}$ and $x > -1$. Prove that $(1+x)^n \geq 1 + nx$ for all natural numbers n.

2. Prove or disprove the following set of asymptotic relations: (3 marks each)

   (a) $(2.9)^{\log_2 n} = \Theta(n^{\log_2 3})$

   (b) $\log\log n = \Omega((\log\log\log n)^{\log\log\log n})$

   (c) $n^4 \sim (1 - 1/n)^n n^3$

   (d) $2^{(\log n) - (\log\log n)} \sim 2^{(1-1/n)\log n}$

   (e) $n^{10(\log\log n)^{100}} = \Theta((\log n)!)$

3. (10 marks) The *Lucas Sequence* $1, 3, 4, 7, 11, 18, 29, \ldots$ is defined by $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$. Prove that $a_n = O(1.75^n)$.

4. (7+7 marks) For natural number p and q, the Ramsey number $R(p, q)$ is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who don't know each other. Prove that Note that $R(p, 1) = R(1, q) = 1$. Prove that:

   (a) $R(p+1, q+1) \leq R(p, q+1) + R(p+1, q)$

   (b) $R(p, q) \leq C_{p-1}^{p+q-2}$

5. (6 Marks) Write the negation of the following statement:

$$\forall x \geq 0 \ \exists y \in \mathbb{N} \ \ (y \geq x) \wedge (y \text{ is a prime})$$

6. (5 + 5 Marks)

   (a) We know that $\sqrt{3}$ is not rational. Using this prove that $\sqrt{3} + \sqrt{24}$ is not rational.

   (b) If m is a positive integers such that $\sqrt{m}$ is not rational then prove that for any positive integer n the number $\sqrt{m} + \sqrt{n}$ is not rational.

7. (10 Marks) In the Towers of Hanoi problem, there are three posts and n disks of different sizes. Each disk has a hole through the center so that it fits on a post. At the start, all n disks are on post #1. The disks are arranged by size so that the smallest is on top and the largest is on the bottom. The goal is to end up with all n disks in the same order, but on a different post. In a single move one can move one disk from one post and place in another post. At no point can one place a bigger disk over a smaller disk in any post. How many steps will be taken to move the n disk from the Post #1 to any other post.

8. (10 Marks) Prove that at a party with at least two people, that there are two people who know the same number of people there (not necessarily the same people - just the same number) given that every person at the party knows at least one person. (Note that nobody can be his or her own friend.)

9. (3+ 7 Marks) A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.

   (a) For any given n, give an example of a tournament which has no directed cycle.

   (b) Prove that a tournament has a directed 3-cycle if and only if it has a directed cycle.

10. (10 marks) How many non-increasing functions be there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.

11. (10 marks) If G is a labeled complete graph, K_n, on n vertices and u, v, w be three distinct vertices in the vertex set of G. How many different paths are there from u to v passing through w.

12. (10 marks) If $G = (V, E)$ is a graph on n vertices such that all the vertices have even degree. Show that the edge set E can be partitioned into pairwise disjoint sets $C_1, C_2, \ldots, C_k$ such that for all $1 \leq i \leq k$ the subgraphs (V, C_i) is a cycle and a collection of isolated vertices.

13. (10 marks) Let c_n^k be the number of ways to distribute n distinguishable balls can be k distinct buckets where the order of balls in a bucket does not count. Set up an ordinary generating function for c_n^k.