**Discrete Mathematics** DATE:24-09-2018

# Lecture 1: Mid sem solutions(1 - 6)

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1. Let x $\in\mathbb{R}$ and $x >$ -1. Prove that $(1+x)^n \geq 1+nx$ for all natural numbers n

Sol: We proceed by induction on n$\in\mathbb{N}$.

Let P( n) be the proposition that (1 + x)$^n \geq 1+nx$ for all natural numbers n, whenever x > -1 , x$\in\mathbb{R}$

Base case: P( 1) is true

Induction Hypothesis: Let P( k) be true for k$\in$ $\mathbb{N}$.

Inductive Step: For n=k+1,

$$(1+x)^{k+1} = (1+x)^k(1+x)$$

$$\geq (1+kx)(1+x) \text{ [By Induction Hypothesis and because } (1+x)\geq 0]$$

$$\geq 1+(k+1)x+kx^2$$

$$\geq 1+(k+1)x \quad as\ x^2 \ \geq \ 0$$

Hence by induction , our propostion is true for all n$\in$ $\mathbb{N}$.

2. Prove or disprove the following set of asymptotic relations:

(a) $(2.9)^{\log_2 n}$ $=\Theta(n^{\log_2 3})$

$(b)$ $\log\log n$ $=\Omega((\log\log\log n)^{\log\log\log n})$

$(c)$ $n^4 \sim (1-\frac{1}{n})^n$

$(d)$ $2^{\log n - \log\log n} \sim 2^{(1-\frac{1}{n})\log n}$

$(e)$ $n^{10(\log\log n)^{100}}$ $=\Theta((\log n)!)$

Sol:

(a) We have $(2.9)^{\log_2 n} = n^{\log_2(2.9)}$. Since $\log_2(2.9) \leq \log_2 3$, so $\frac{(2.9)^{\log_2 n}}{n^{\log_2 3}} = n^{\log_2(2.9)-\log_2 3}$ which tend to 0 as $n$ goes to infinity. So $(2.9)^{\log_2 n} = o(n^{\log_2 3})$ and hence the given statement is not true

(b) Let's take, x = $\log\log\log n$.

Now, $$\text{e}^x = o(x^x) \qquad \forall \quad \text{increasing function } x.$$

$$\therefore \qquad \log\log n = O((\log\log\log n)^{\log\log\log n})$$

, disproving the given statement.

(c) $\lim_{n\to\infty}$ (1-$\frac{1}{n}$)$^n$ $= e^{\text{-}1}$ while $\lim_{n\to\infty}$ n is divergent.Thus we disprove the given statement.

(d) Let $x = \log n - \log\log n$ and $y = (1 - \frac{1}{n})\log n$.

If the statement were true, $\lim_{n\to\infty} 2^x/2^y = 1$

$$\text{i.e. } \lim_{n\to\infty} 2^{x-y} = 1$$

$$\text{i.e. } \lim_{n\to\infty} x - y = 0.$$

But $x - y = \frac{\log n}{n} - \log\log n$. As $n \to \infty$, $\frac{\log n}{n} \to 0$ and $\log\log n \to \infty$. So $\lim_{n\to\infty} x - y \neq 0$ and hence the statement is false.

(e) For the given statement to be true, $\exists\ c \ni\ n^{10(\log\log n)^{100}} \leq\ c(\log n)!\ for\ all\ n\ > N_0\ for\ some\ N_0 \in \mathbb{N}$.

Also since $(logn)! < (logn)^{\log n}$ so for the given statement to be true,$\exists\ c \ni\ n^{10(\log\log n)^{100}} \leq\ c(\log n)^{\log n}\ for\ all\ n\ > N_0\ for\ some\ N_0 \in \mathbb{N}$

$$\text{But } \lim_{n\to\infty} \frac{n^{10(\log\log n)^{100}}}{(\log n)^{\log n}} = \infty$$

So, no such c exists. The given statement is false.

3. The Lucas Sequence 1, 3, 4, 7, 11, 18, 29, . . . is defined by $a_1 = 1, a_2 = 3, a_n = a_{n\text{-}1} + a_{n-2}$.

Prove that $a_n = O(1.75n)$.

Sol: Given $a_n = a_{n\text{-}1} + a_{n-2}, \forall n \geq 3, a_1 = 1, a_2 = 3$.

Lets use generating functions to solve it.

Define, $A(x)=\sum_{i=1}^{\infty} a_i x^i$.......$(i)$

$$=x+3x^2 + \sum_{i=3}^{\infty} a_i x^i$$

$$=x+3x^2 + \sum_{i=1}^{\infty}(a_{i\text{-}1} + a_{i-2})x^i$$

$$=x+2x^2 + x\sum_{i=1}^{\infty} a_i x^i + x^2\sum_{i=1}^{\infty} a_i x^i$$

$$=x+2x^2 + xA(x) + x^2A(x)$$

Therefore, A(x)= - $\frac{x+2x^2}{1-x-x^2}$.....(ii)

The roots of the denominator are $\alpha = -\frac{(1+\sqrt{5})}{2}$ and $\beta = \frac{(\text{-}1+\sqrt{5})}{2}$

We can write A(x) as A(x)= -x($\frac{\gamma}{x-\alpha}$+$\frac{\delta}{x-\beta}$).....(iii)

From (ii) & (iii) we get $\gamma = 1, \delta = 1$,

$$
\begin{aligned}
\therefore \qquad A(x) &= -x(\tfrac{1}{x-\alpha} + \tfrac{\delta}{x-\beta}) \\
&= \text{-x}\{\tfrac{1}{\alpha(-\beta x\text{-}1)} + \tfrac{1}{\beta(-\alpha x\text{-}1)}\} \quad [\because \alpha\beta = 1] \\
&= \text{x}\{\tfrac{1}{\alpha(\beta x+1)} + \tfrac{1}{\beta(\alpha x+1)}\} \\
&= \text{x}\{\tfrac{1-x\beta+x^2\beta^2-x^3\beta^3+.....}{\alpha} + \tfrac{1-x\beta+x^2\beta^2-x^3\beta^3+.....}{\beta}\}......\text{(iv)} \ ,
\end{aligned}
$$

using bionomial expansion of $(1+\text{x})^{-1}$

Equating (i) & (iv), we get

$$
\begin{aligned}
\text{a}_n &= \tfrac{(-\beta)^n}{\alpha} + \tfrac{(-\alpha)^n}{\beta} \\
&= \tfrac{(1-\sqrt{5})^n+(1+\sqrt{5})^n}{2^n} \\
&< \tfrac{(-1.236)^n+(3.236)^n}{2^n} \\
&< (0.618)^n + (1.618)^n \\
&< 2(1.618)^n \\
&< 2(1.75)^n
\end{aligned}
$$

Hence, $\text{a}_n = O((1.75)^n)$

Alternatively,

Given $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$, we shall show, $a_n \leq c(1.75)^n$

We proceed by induction on $n$.

Base Case : $a_1 \leq c.1$ $a_2 \leq 2c.(1.75)^2$ valid for $c \geq 1$.

Inductive hypothesis : We assume, $a_i \leq c(1.75)^i$ For $i \leq c(1.75)^i$ for $i \leq n-1$

$$
\begin{aligned}
\text{a}_n &= a_{n-1} + a_{n-2} \\
&\leq c(1.75)^{n-1} + c(1.75)^{n-2} \\
&= \text{c}(1.75)^{n-2}(1 + 1.75) \\
&= \text{c}(1.75)^{n-2}(2.75) \\
&< \text{c}(1.75)^n \quad for\ c > 1.
\end{aligned}
$$

Hence, $\text{a}_n = O((1.75)^n)$

4. For natural number p and q, the Ramsey number R(p, q) is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who dont know each other. Note that R(p, 1) = R(1, q) = 1. Prove that:

(a) R(p + 1, q + 1) $\leq R(p, q+1) + R(p+1, q)$

(b) R(p, q) $\leq \binom{p+q-2}{p-1}$

Sol: Let us observe the problem in terms of graph theory. We denote G(V; E) an undirected graph as follows,

$$V=\{v \mid v\ is\ a\ person\}$$

$$E=\{(u,v) \mid u\ knows\ v\}$$

Then by definition, R(p,q) is the smallest natural number n such that a graph with n vertices has either a size-p clique or a size-q independent set.

(a) Let $$R(p+1,q) + R(p,q+1) = n$$

We need to prove that for any graph G on n vertices, G either has a size-(p + 1) clique or a size-(q + 1) independent set. Let us choose any vertex v $\in V$. *We define*,

$$N_v = \{u \in V \mid (u,v) \in E\}$$

$$\overline{N_v} = \{v \in V \mid (u,v) \notin V\}$$

Now either $\mid N_v \mid \geq R(p,q+1)\ or\ \mid \overline{N_v} \mid \geq R(p+1,q).\ Because\ if\ not\ then$

$$\begin{aligned} \mid V \mid &= \mid N_v \mid + \mid \overline{N_v} \mid + \mid \{v\} \mid \\ &\leq R(p,q+1) - 1 + R(p+1,q) - 1 + 1. \\ &< n \end{aligned}$$

a contradiction.

CASE I

If $\mid N_v \mid \geq R(p,q{+}1)$ then the induced subgraph of G on the vertex set $N_v$ has a size-p clique or a size-(q+1) dependent set. Thus this subgraph along with vertex v has either a size(p+1)- clique or a size-(q+1) dependent set. Hence G will have either a size-(p+1) clique or a size-(q+1) independent set.

CASE II

If $\mid \overline{N_v} \mid \geq R(p{+}1,q)$ then the induced subgraph of G on the vertex set $N_v$ has a size-(p+1) clique or a size-q dependent set. This subgraph along with vertex v has either a size-(p+1) clique or a size-(q+1) dependent set.

Hence G will have either a size-(p+1) clique or a size-(q+1) independent set.

Hence we have proved by exclusive case studies that

$$R(p+1, q+1) \leq R(p,q+1) + R(p+1,q)$$

(b) We prove the given inequality by induction on p and q .Let

$$R(i,q) \leq \binom{i+q-2}{i-1}$$

$$R(p,i) \leq \binom{p+i-2}{p-1}$$

Base Case: R(1,q)=1 $\forall p \in \mathbb{N}$

R(p,1)=1 $\forall q \in \mathbb{N}$

Lets assume that the given inequality holds for p=k and q=l, and prove it for p=k+1 and q=l+1.

$$\begin{aligned} \text{R(k+1,l+1)} &\leq\ R(k,l+1)\ +\ R(k+1,l) \quad from\ 4(a) \\ &\leq\ \binom{k+l-1}{k-1}\ +\ \binom{k+l-1}{k} \\ &= \binom{k+l}{k} \quad [\because \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}] \end{aligned}$$

Hence, by induction, the given inequality is true.

5. Write the negation of the following statement: $\forall x \leq 0\ \exists\ y \in \mathbb{N} \quad (y \geq x) \wedge$ (y is a prime)

Sol: $\neg$ ( $\forall x \leq 0\ \exists\ y \in \mathbb{N} \quad (y \geq x) \wedge$ (y is a prime))

$=\ \exists\ x \geq 0\ ,\ \forall y \in \mathbb{N}, \neg\ \{(y \geq x) \wedge (y\ is\ a\ prime)\}$

$=\ \exists\ x \geq 0\ ,\ \forall y \in \mathbb{N},\ (y \leq x) \vee (y\ is\ a\ prime)$

6.

(a) We know that $\sqrt{3}\ is\ not\ rational.\ Using\ this\ prove\ that\ \sqrt{3}\ +\ \sqrt{24}\ is\ not\ rational.$

(b) If m is a positive integers such that $\sqrt{m}$ is not rational, then prove that for any positive integer n the number $\sqrt{m} + \sqrt{n}\ is\ not\ rational.$

Sol:

(a) Let's assume that $\sqrt{3} + \sqrt{24}\ is\ rational.\ Then,$

$$\sqrt{3} + \sqrt{24} = \tfrac{p}{q} \quad \text{,p,q} \in\ \mathbb{Z}$$

or $$\text{q}\ \sqrt{24} = p - \sqrt{3}q$$

or $$24\text{q}^2 = (p - \sqrt{3}q)^2$$

or $$24\text{q}^2 = p^2 + 3q^2 - 2\sqrt{3}pq$$

or $$\sqrt{3} = \tfrac{p^2-21q^2}{2pq} \quad which\ is\ rationl.$$

But $\sqrt{3}\ is\ not\ rational.$ Thus we get a contradiction.

(b) Given, $\sqrt{m}\ is\ not\ rational,\ weassume,\ if\ possible,\ that \sqrt{m} + \sqrt{n}\ is\ rational. Then$

$$\sqrt{m} + \sqrt{n} = \tfrac{p}{q}\ where\ p, q\ \in\ \mathbb{Z}$$

or $$\sqrt{n}q = p - \sqrt{m}q$$

or $$\text{m}^2q^2 - 2\sqrt{m}pq + p^2 = nq^2$$

or $$\sqrt{m} = \tfrac{p^2-21q^2}{2pq}$$ which shows that $\sqrt{m}$ is rational, a contradiction.

Alternatively,

Assume for the sake of contradiction that $\sqrt{m} + \sqrt{n}$ *is rational*

$$(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n}) = m - n.$$

This shows that $(\sqrt{m}-\sqrt{n})$ is rational, from the fact that a rational no. multiplied by an irrational no. cannot give a rational no. Now, if $(\sqrt{m}+\sqrt{n})$ *and* $(\sqrt{m}-\sqrt{n})$ *are both rationals, their sum,* $2\sqrt{m}$ *must also be* rational, a contradiction.