Discrete Mathematics 01/10/2018

Mid Semester Examination Solutions Q7-Q13

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Q7. In the Towers of Hanoi problem, there are three posts and n disks of different sizes. Each disk has a hole through the center so that it fits on a post. At the start, all n disks are on post 1. The disks are arranged by size so that the smallest is on top and the largest is on the bottom. The goal is to end up with all n disks in the same order, but on a different post. In a single move one can move one disk from one post and place in another post. At no point can one place a bigger disk over a smaller disk in any post. How many steps will be taken to move the n disk from the Post 1 to any other post.

Solution:

The towers of Hanoi problem can be solved recursively as follows:

*Step*1: Move top (n-1) disks to tower 2,via tower 3.

*Step*2: Move the n^{th} disk to tower 3.

*Step*3: Move the (n-1) disks from tower 2 to tower 3 , via tower 1.

If S(n) be the number of steps to move all the disks from tower 1 to tower 3 , we have :

*Step*1 :S(n-1) ; *Step*2 :1 ; *Step*3 :S(n-1) steps

$\therefore$ S(n)=2S(n-1)+1

Claim: S(n)=2^n - 1

Pf:

2S(n-1)+1= 2 $(2^{n-1} - 1)$ + 1 = 2^n-1 = S(n)

Hence , the number of steps taken are $2^n - 1$

Q8. Prove that at a party with at least two people, that there are two people who know the same number of people there (not necessarily the same people - just the same number) given that every person at the party knows at least one person. (Note that nobody can be his or her own friend.)

Solution:

Let us assume that in a party with n people, each people knowing at least one person, there are no such pair of persons who know the same number of people

Suppose person p_i knows i number of people. Since the value of i varies with each different person, we say that each number of people know unique number of other people

p_1 knows one person , p_2 knows two person .... p_n knows n persons.

However this is not possible, since the maximum number of persons a person can know is $n-1$

Therefore P_n must know a number of person $< n$, say equal to k ; $k < n$ But p_k also knows k number of people. Hence, $\Longrightarrow\Longleftarrow$

Therefore at least two people know equal number of people

Please note that this can be directly attained by using the Pigeon Hole Principle.

Q9. A tournament is a directed graph (digraph) obtained by assigning a

direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge. (a) For any given n, give an example of a tournament which has no directed cycle. (b) Prove that a tournament has a directed 3-cycle if and only if it has a directed cycle.
Solution:
Convention used in the solution : In a directed graph, if a and b are vertices, (a,b) implies there is a directed edge from a to b , pointing from a , towards b

a)Consider the tournament of three vertices a,b,c with the edges (a,b) , (b,c), (a,c) . This is a valid tournament without a directed cycle.
b) Proof by induction for part 1:
If there is a directed cycle, it implies that a 3 cycle is present
Induction on the number vertices in the cycle , or the length of the cycle n
Base case : For n=3 , we trivially satisfy the condition.
Let this hypothesis be true for n=k [Weak Induction]
Now consider a tournament of k+1 vertices that form a cycle. For a specific vertex v_i, we have an edge (v_{i-1},v_i) and an edge (v_i,v_{i+1}). There is also an edge from v_{i-1} to v_{i+1} either in the form of (v_{i+1},v_{i-1}) or (v_{i-1},v_{i+1}). We encounter two cases here:
i)The edge is (v_{i+1}, v_{i-1}). We have a three cycle in this case: (v_{i-1}, v_i) , (v_i, v_{i+1}), (v_{i+1}, v_{i-1}).
ii)The edge is (v_{i-1}, v_{i+1}). We remove v_i and all edges incident on it. Note that this means that we have a directed cycle of length k now, which , by our assumption contains a 3 cycle. Clearly this 3 cycle has no involvement of v_i , so if we add back v_i now, this 3 cycle is still intact. Hence our assumption holds for n=k+1.
Thus proved.
Proof for part 2:
If there is a 3 cycle, the tournament has a directed cycle
Trivially proved

Q10. How many non-increasing functions be there from {1, 2, . . . , n} to {1, 2, . . . , k}
Solution:
There is exactly one descending arrangement of the range B (let) = {1,2,3...,k}
Now if we have a non increasing function and $f(a_1) = b, f(a_2) = b, f(a_3) = b...$, then a_1, a_2, a_3 are consecutive. This means that we can have a partition of the domain of n elements into k parts , each partition corresponding to a group of values in the domain, mapping to a single element in the range.
Number of ways to do this is $\binom{n+k-1}{n}$

Q11. If G is a labeled complete graph, Kn, on n vertices and u, v, w be three distinct vertices in the vertex set of G. How many different paths are there from u to v passing through w
Solution:
We notice the length of the path from u to v ; via w. We notice that the length is a minimum of 2 , and a maximum of n , i.e. all vertices taken into account.

Now if there are i intermediate vertices , we have $\binom{n-3}{i}i!$ ways to arrange these vertices. Then we have (i+1) ways to put place the vertex w between u and v.
$\therefore$ the number of ways we can have i intermediate vertices are :
$(i+1)!\binom{n-3}{i}$

Summing over i from 0 to n-3 to get the total number of paths, we have :
$\sum_{i=0}^{n-3}(i+1)!\binom{n-3}{i}$
Since we are dealing with permutations here, finding a closed form is difficult.
The results of this sequence can be obtained from : http://oeis.org/A001339

Q.12 If G = (V, E) is a graph on n vertices such that all the vertices have even degree. Show that the edge set E can be partitioned into pairwise disjoint sets C1, C2, . . . , Ck such that for all 1 i k the subgraphs (V, Ci) is a cycle and a collection of isolated vertices.
Solution:
If all vertices in a graph has even degree, then there must be a cycle(This has been solved in class , with a contradiction argument, considering the longest path that has all vertices of even degree , but no cycle; The last vertex violated the even degree property) . Remove this cycle. The resulting graph must have all vertices to have even degree always. This means the reduced graph also has a cycle, or it has all vertices of degree 0. Continue this till we are left with vertices of only zero degree.
Hence we have partioned the graph into disjoint cycles and isolated vertices

Q13. Let c_k^n be the number of ways to distribute n distinguishable balls can be k distinct buckets where the order of balls in a bucket does not count. Set up an ordinary generating function for c_k^n.
Solution:
For n distinct balls, the number of ways to place them in k distinct buckets is : k^n
The generating function should have a 1-1 correspondence of its coefficients and the value of the number of ways.
Let us define $g(n) = k^n/n!$
Then $\sum_{n=0}^{\infty} g(n)x^n = \sum_{n=0}^{\infty}(kx)^n/n!$
Coefficient of each term of this function is $k^n/n!$,for $n = 1, 2, ...$ Hence it maintains a 1-1 correspondence with k^n
Hence the generating function is:
$\sum_{n=0}^{\infty}(kx)^n/n! = e^{kx}$