Discrete Mathematics 30/08/2018

Quiz 1: Solutions

Instructor: Sourav Chakraborty *Scribe: Sourav Saha*

1 What is the negation of the following statement: "For all C, D, E, F ≥ 0 there exists an n $\in \mathbb{N}$ such that for all n $\geq$ N we have C.$2^n >$ Dn^8 and E(log)$^4 <$ F(n$^{1/100}$)"

The given statement can be written as $\forall$ x $\exists$ y $\forall$ z P(x,y,z) where x is "C,D,E,F $\geq$ 0", y is "n $\in \mathbb{N}$" and z is "n $>$ N".

So, we can rewrite it as $\neg$($\forall$ x $\exists$ y $\forall$ z P(x,y,z)) $\Longrightarrow$ ($\exists$ x $\forall$ y $\exists$ z $\neg$ P(x,y,z)). Therefore, the negation of the given statement is : There exists C,D,E,F $\geq$0 such that for all n$\in \mathbb{N}$ some n$>$ N such that C.$2^n \leq$ D$n^8 or E(log)^4 \geq$ F(n$^{1/100}$)).

2 Prove that a graph is bipartite if and only if the graph has no odd cycle.

The proof of the above statement has two parts:
1. Necessity condition : Graph is bipartite $\Longrightarrow$ Graph has no odd cycle.
Let G be a bipartite graph. If we take any walk in the graph, it alternates between the two sets in the bi partition. So, to make a complete cycle it needs to come back to the original partition set from where it started the walk. As there is a bi-partition it can come after even number of steps only. Therefore we can conclude that G has no odd cycle.

2. Sufficient Condition : Graph has no odd cycle $\Longrightarrow$ Graph is bipartite.
Suppose that G has no odd cycle. Let us choose any vertex v $\in G$. We can construct two sets of vertices say V1 and V2 such that,
V1 is the set where shortest path from element i to v is even(i $\in$ V1)
V2 is the set where shortest path from elment j to v is odd(j $\in$ V2)
where v belongs to G.
Therefore the intersection of the above sets V1 and V2 is empty. No two vertices in V1 or V2 are adjacent as the graph G does not contain any odd cycle.
So, V1 and V2 are two partition from V.
We can conclude that the graph has no odd cycle implies graph is bipartite. (only if case).
Hence a graph is bipartite iff the graph has no odd cycle.

3 If T(n) = 3T($\lceil n/3 \rceil$) + 1 and T(1) = 1 then what is T(n). (Either the exact expression or the closest asymptotic expression)

Let's assume n = 3^k.

$$
\begin{aligned}
T(n) &= 3T(n/3) + 1 \\
&= 33T(n/9) + 1 + 1 \\
&. \\
&. \\
&. \\
&= 3^kT(n/3^k) + 3^(k-1) + 3^(k-2) + ... + 1 \\
&= 3^kT(1) + (3^k - 1)/2 \leq \text{cn}
\end{aligned}
$$

Let's guess T(n) $\leq$ cn - d.

We shall prove by induction that T(n) $\leq$ cn - d for some constants c and d to be decided later.

Base Case: T(1) = 1 $\leq$ c - d holds true.

Inductive Step: Let us induct on n

$$
\begin{aligned}
T(n) &= 3T(\lceil n/3 \rceil) + 1 \\
&\leq 3(\text{cn}/3 - \text{d}) + 1 \\
&= cn - 3d + 1 \\
&\leq \text{cn - d (for some d} > 1/2)
\end{aligned}
$$

Therefore T(n) = O(n).

4 Prove that if a graph has at most m vertices of degree at most n and all other vertices have degree of at most k, with k < n and m < n, then the graph is (vertex) colorable with m+k+1 colors.

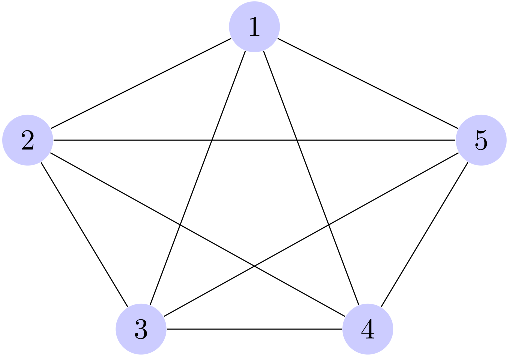
Base Case: Consider m = 0, which implies every vertices have degree at most k. So we can color the vertices with k+1 colors.

Inductive Hypothesis: For all those graphs have at most m vertices of degree at most n and all other vertices have degree of at most k, with k < n and m < n, the graph is vertex colorable with m+k+1 colors implies. For all such graphs have at most m+1 vertices of degree at most n and all other vertices have degree of at most k, with k < n and m+1 < n, the graph is vertex colorable with m+k+2 colors.

Inductive Step: Let us consider a graph G with m+1 vertices of degree at most n and all other vertices having degree at most k. Pick any of these m+1 vertices V that has degree at most n and delete it from G. So, the resulting graph G' = G/v is by induction hypothesis m+k+1 colorable. Now lets add another vertex v' that has degree of at most n. The resulting graph G" has m+1 vertices of degree n and using a new color C(say) for the vertex v' other than those m+k+1 colors, we can color the resulting graph G" in m+k+2 colors. Therefore, by method of induction we proof that if a graph has at most m vertices of degree at most n and all other vertices have degree of at most k, with k < n and m < n, then the graph is (vertex) colorable with m+k+1 colors.

5 If n ≡ 1(mod 4) then prove that there is a way of coloring the edges of K_n with colors red or blue such that for all vertex exactly half of the adjacent edges is colored red and the other half is colored blue.

Base Case : For K = 5 (below graph as shown) we can color v_1-v_2, v_2-v_3, v_3-v_4, v_4-v_5, v_5-v_1 with red colors and rest of the edges with blue colors. So, for all vertices exactly half of the adjacent edges are colored with red and other half is colored with blue.



Inductive Hypothesis: For n ≡ 1(mod 4) there is a way of coloring the edges of K_n with colors red or blue such that for all vertex exactly half of the adjacent edges is colored red and the other half is colored blue. For all such graph of K_{n+4} there is a way of coloring the edges with colors red or blue so that for all vertex exactly half of the adjacent edges is colored red and the other half is colored blue.

Inductive Steps: Let us induct on the number of vertices n. So K_n holds true with colors red or blue such that all vertex exactly half of the adjacent edges is colored red and the other half is colored blue. We need show that it is valid for K_{n+4}. Let's add 4 vertices in the K_{n+4} graph. Now pull out one vertex from K_n (say v_0) and we add four vertices say v_1, v_2, v_3, v_4. We observe the coloring of the vertex v_0 and colored the vertex v_1 and v_2 with the same coloring and v_3 and v_4 with the complement of the color from vertex v_0. So, we defined the coloring C of the new 4 vertices in this way:

C(v1, w) = C(v0, w), C(v2, w) = C(v0, w), C(v3, w) = C'(v0, w), C(v4, w) = C'(v0, w) ∀ w ∈ K_n For v_0 half of the edges are colored with red and other half are colored with blue. Complement of the color of the edges of v_0 is also valid, therefore for 4 vertices v_1, v_2, v_3, v_4 exactly half of the edges are colored with red and other half is colored with blue. We are now left with coloring of the edges between v_0,v_1,v_2,v_3,v_4. We can color the edges as the same way we did the base case i.e K_5. So, we can conclude that K_{n+4} is holds true i.e exactly half of the color of the edges of its adjacent edges can be colored with blue and other half can be colored with red.