Discrete Mathematics	10 Nov 2018

Quiz-2

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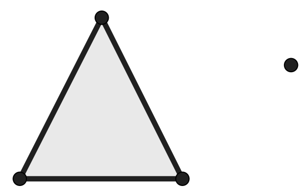
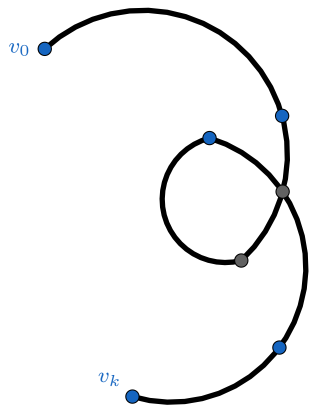
1. If G is a graph with even number of vertices and each vertex has even degree (that is, even number of neighbors) then G has even number of edges. Either prove the statement or disprove it by demonstrating a counter example.
Solution:



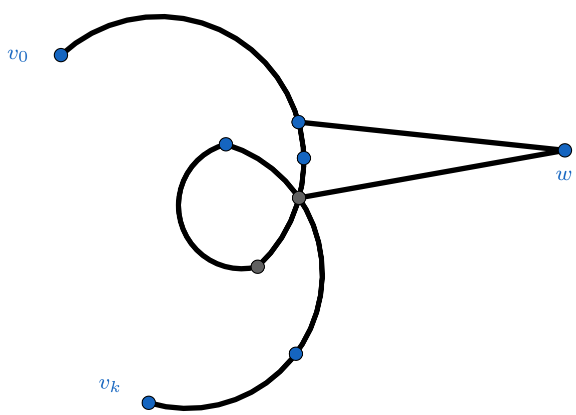
Number of vertices of the graph is 4 but number of edges is 3 which is odd. Hence we get a contradiction.

2. Is it possible to draw 9 line segments in the plane $\mathbb{R}^2$ such that each line segment intersects exactly 3 other line segments.
Solution:
Consider each line segment as a vertex of a graph G and draw an edge between each vertex whose corresponding line segment intersects each other.
Then $deg(v) = 3 \; \forall v \in G$. Then $|l| = \frac{3\times9}{2} \notin \mathbb{Z}$ where $|l|$ denotes number of edges. Hence the above statement cannot be possible.

3. Prove that a graph has an Eulearian path if and only if the number of vertices in the graph with odd degree is 0 or 2.
Solution:
First part is fairly straightforward.



Let $v_0, \ldots, v_k$ be an Euler trail. So each v_i has even degree except v_0 and v_k.If this is an Euler trail then v_0 and v_k are two endpoints; hence number of odd degree vertices are two. If $v_0 = v_k$ then it is an Euler tour. There the number of odd degree vertices are zero.



Conversely, let l be the longest path in G. There are atmost two odd degree vertices. Let $v_0, \ldots, v_k$ be the path. v_k is not adjacent to any other vertex because if it was the case then l will not be the longest path.
Let l be not Eulerian then there exist some path v_i, v_{i+1} which are not part of l because each v_i is of even degree So there exist another path which is also not part of l.If G has zero odd degree vertex then $v_k = v_0$. Now if we start a path from v_i then traverse l that will be longer than l. (Contradiction)
So, l is Eulerian.
Now let G has 2 odd degree vertices say u and v. Add new vertex w. Join uw and vw. So degree of w is 2. So degree of v and u is even.
So $H = G \cup w$. So H has all even vertices of even degree. So H has an Euler tour. Choose an Euler tour that start and end in w. If we remove w from the Euler tour we get Euler trail. Hence proved.

4. Consider an $n \times m$ rectangular grid, with the co-ordinates of the corners being

$(0,0),(n,0),(n,m)$ and $(0,m)$. How many paths are there along the rectangular grid from $(0,0)$ to (n,m) such that

(a) The paths are shortest among all paths from $(0,0)$ to (n,m).
(b) Every horizontal move of unit length is followed by at least one vertical move and the paths are the shortest.
Solution:
(a) The number of shortest paths is $\binom{n+m}{n}$.

6. For a simple graph G an independent set is a subset S of vertices such that no two vertices in the set S are adjacent. Prove that

(a) If a graph on n vertices has a matching of size k then G cannot have an independent set of size $> (n-k)$.
(b) If the graph G on n vertices has a maximal matching of size k then G has an independent set of size $\geq (n-2k)$.
Soluton:
(a) Let there be a matching of size k which is forming a bipartite graph component X and Y. So there are $n-2k$ vertices remaining in G. Let S be the set consisting of those remaining vertices of G. This set is independent.
Let S is only adjacent to Y. So $S+X=S'$ is also independent.
$|S'| = n-2k+k = n-k$
But we cannot add any vertex to s because all other vertices are adjacent to S. Hence, G cannot have independent set of size $> n-k$.
(b) Let S be a maximal matching of size k. S' be a set consists of remaining $n-2k$ vertices. If this set is not independent, then there exists some edge between vertices v_1, v_2 which belongs to S', so then we extend the size of S. But S is maximal hence a contradiction.