Discrete Mathematics 08/11/2018

Quiz 2b Solutions

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Q.1 If G is a connected graph with even number of vertices and each vertex has even degree (that is, even number of neighbors) then G has even number of edges. Either prove the statement or disprove it by demonstrating a counter example.

Solution:

Consider the example:

It has even number of vertices(6) and degree of each vertex is even.

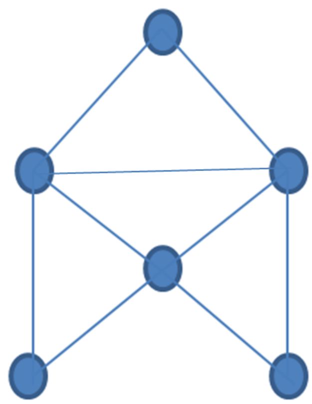

Figure 1:

Degree Sequence $=\{4,4,4,2,2,2\}$

Total edges $= 9$

Here number of edges is not necessarily even.

$\therefore$ *Disproved.*

Q.2 Maria and her partner organize a party together with 4 other couples. There are a number of greetings but, naturally, nobody says hello to their own partner. At the end of the party Maria asks everyone how many people did they greet, and she receives nine different answers. How many people did Maria greet and how many people did her partner greet?

Solution:

Given:

The party has 5 couples and no couple can greet each other.

So for each person in the party there are 8 different people who he/she can greet.

Maria receives 9 different answers on asking everyone.So the only possibility of that happening is (0,1,2,3,4,5,6,7,8) greetings.

Let degree of Maria be x and so list of degrees is

(8 7 6 5 4 3 2 1 0 x)

Now by Havel-Hakimi theorem:
1. Remove 8 degree and then subtract 1 from the next eight terms to get
$(\not 8\ 6\ 5\ 4\ 3\ 2\ 1\ 0\ \not 0\ x\text{-}1)$
8 - 0 are a couple
2. Remove 6 degree and then subtract 1 from the next six terms to get
$(\not 8\ \not 6\ 4\ 3\ 2\ 1\ 0\ \not 0\ \not 0\ x\text{-}2)$
7 - 1 are a couple
3. Remove 4 degree and then subtract 1 from the next 4 terms to get
$(\not 8\ \not 6\ \not 4\ 2\ 1\ 0\ \not 0\ \not 0\ \not 0\ x\text{-}3)$
6 - 2 are a couple
4. Remove 2 degree and then subtract 1 from the next 2 terms to get
$(\not 8\ \not 6\ \not 4\ \not 2\ 0\ \not 0\ \not 0\ \not 0\ \not 0\ x\text{-}4)$
5 - 3 are a couple
According to Havel-Hakimi we stop when all are zeroes, hence $x-4 = 0 \Rightarrow x=4$
That is Maria greeted 4 people and coupled with only remaining 4 degree vertex i.e. partner also greeted 4 people.

Q.3 Prove that a graph has an Eulerian path if and only if the number of vertices in the graph with odd degree is 0 or 2.
Solution:
To Prove:
A graph with Eulerian Path $\Leftrightarrow$ Number of degree is 0 or 2.

To Show:
A graph with Eulerian Path $\Rightarrow$ Number of odd degree is 0 or 2
Proof:
For a Eulerian path to exist every edge must be traversed exactly once.
So all the vertices in the center of the path must have equal indegree and outdegree, as every time a path reaches a vertex, it will need one inlet and one outlet which has not been used up yet.
The ends of the path can have 1 less inlet/oulet. So number of odd degree is 0 or 2.
If it doesnt have any odd degree, we can form a Eulerian cycle by returning to first vertex.
So if there are equal number of inlets and outlet in the centre i.e. all middle vertices have even degree, $\exists$ an Eulerian path.
If there is only one odd degree then edge will be left untraced.
So if number of odd degree 0 or 2 $\Leftrightarrow$ Eulerian Path.

Q.4 Consider a n m rectangular grid, with the co-ordinates of the corners being (0, 0), (n, 0), (n, m) and (0, m) . How many paths are there along the rectangular grid from (0, 0) to (n, m) such that
(a) The paths are of length k but they need not be the shortest path.
Solution:
Given, grid of n x m and length of path is k.
The shortest path is of length (m+n).
So if k<(m+n), number of paths = 0.

If k=(m+n); number of shortest path are $\frac{(m+n)!}{n!m!}$.
If k>m+n; then there are additional movements such as left and down in addition to up and right.
In $k-(m+n)$ extra steps there has to be additional right for every left and additional up for every down.
If $k-(m+n)$ is Odd; number of paths = 0.
If $k-(m+n)$ is Even; let a represents number of left and b represent number of down then $k-(m+n)=2a+2b$.
So number of paths = $\frac{k!}{(n+a)!a!(m+b)!b!}$

$$Numberofpaths = \begin{cases} 0, & \text{if } k < m+n \\ \binom{m+n}{n}, & \text{if } k = m+n. \\ \sum_{\{a,b|a+b=\frac{k-(m+n)}{2}, a>=0, b>=0\}} \frac{k!}{(n+a)!a!(m+b)!b!}, & \text{,if } k > m+n \text{ and } k-(m+n) \text{ is Even.} \end{cases} \tag{1}$$

(b) There are no two consecutive horizontal moves and the paths are the shortest.
Solution:
The shortest path is of length (m+n).
If $n > m$; number of paths = 0.
If $n <= m$; it is similar to choosing n places for horizontal moves between m vertical moves.

$$Numberofpaths = \begin{cases} 0, & \text{if } n > m \\ \binom{m+1}{n}, & \text{if } n <= m. \end{cases} \tag{2}$$

Q.5 Prove: If the graph G has no simple path of length k then G is k colorable
Solution:
To Show: Graph having longest simple path of length k-1 is k colorable.
Proof:
k-1 length path has k vertices, and each vertex could only connect to other k-1 vertices, therefor maximum possible degree of a vertex is k-1.
Now we have to show graph having vertices with maximum possible degree of k-1 is k colorable.
Proof by Induction:
We use induction on the number of vertices in the graph.
Base Case:. A 1 vertex graph has maximum degree 0 and is 1 colorable, so P(1) is true.
Induction Hypothesis: P(n) is true that is a graph with n vertices with maximum degree atmost (k-1) is k colorable.
Inductive Step: Let G be an (n + 1)vertex graph with maximum degree at most (k-1).Remove a vertex v, leaving an n vertex graph G'.The maximum degree of G' is at most k-1, and so G' is k colorable by our assumption P(n). Now add back vertex v. We can

assign v a color different from all adjacent vertices, since v has degree at most (k -1) and k colors are available. Therefore, G is (k) colorable.
Hence, Graph having longest simple path of length k-1 is k colorable.

Q.6 Let G be a graph with $|E| = |V| - 1$ that is not a tree.
(a) Prove that G has at least one connected component that is a tree and at least one that is not a tree.
Solution:
To Show: G has atleast one component tree
Proof by Contradiction:
Assume that all connected components are not trees, then each component must have edges $> |V| - 1$.
Assume we have n such connected components: $V_1, V_2, ..., V_n$ such that $|V| = |V_1| + |V_2| + ... + |V_n|$, then number of edges in entire graph will be sum of edges of individual components, i.e. $|E| = |E_1| + |E_2| + ... + |E_n|$.
And as each component is not a tree, then $|E_1| >= |V_1|, |E_2| >= |V_2|, ..., |E_n| >= |V_n|$
So total edges $|E| >= |V_1| + |V_2| + ... + |V_n| = |V|$, which contradicts $|E| = |V| - 1$. Therefore, G has atleat one component as tree.

To Show: G has atleast one component that is not a tree.
Proof by Contradiction:
Assume that all connected components are trees, then each component must have edges $= |V| - 1$.
Assume we have n such connected trees: $V_1, V_2, ..., V_n$ such that $|V| = |V_1| + |V_2| + ... + |V_n|$, then number of edges in entire graph will be sum of edges of individual trees, i.e. $|E| = |E_1| + |E_2| + ... + |E_n|$.
And as each component is a tree, then $|E_1| = |V_1| - 1, |E_2| = |V_2| - 1, ..., |E_n| = |V_n| - 1$
So total edges $|E| = |V_1| + |V_2| + ... + |V_n| - n$, which contradicts $|E| = |V| - 1$. Therefore, G has atleast one component as not a tree.
So G has at least one connected component that is a tree and at least one that is not a tree.

(b) Prove that if G has exactly two connected components, then the one that is not a tree has exactly one cycle.
Solution:
Let two components be V_1, V_2 with edges E_1, E_2 such that,
$|V| = |V_1| + |V_2|$
$|E| = |E_1| + |E_2|$
WLOG, consider V_1 to be tree then, $|E_1| = |V_1| - 1$
So, $|V| - 1 = |V_1| - 1 + |E_2| \Rightarrow |E_2| = |V_2|$.
Now for V_2 to be a tree ,required number of edges is $|E_2| - 1$, that means V_2 has 1 extra edge which prevents it from being a tree and being a connected component that extra edge has to be between 2 already connected vertices which makes 1 cycle.
So V_2 has exactly 1 cycle.