Discrete Mathematics	12/09/2018

Lecture 11: Recurrence: Derangement

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1 Derangement

1.1 Principle of Inclusion and Exclusion

Suppose there are n letters and n envelopes:
L_1,L_2,............,L_n
E_1,E_2,............,E_n
such that letters go to the correct envelope if L_i goes to E_i.

So in how many ways can we put the letters into envelope such that no letter goes to the right envelope?

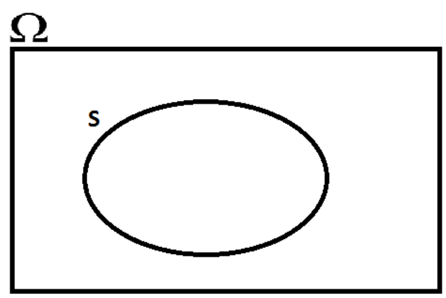

Figure 1: Fig. representing the set S in its universal set Ω.

Ω : set of all possible ways of putting n letters to n envelopes.
So, $|\Omega|$ = n!
S : set of all letters has been put wrongly.
$\bar{S}$: set of at least one letters has been put correctly.
There are two approaches we can proceed: count the set S or count the set $\bar{S}$.
It would be convenient to count the set $\bar{S}$
Now if we split the set $\bar{S}$ into different sub-cases:

1. Exactly 1 letter put correctly represented by set T_1.
2. Exactly 2 letters put correctly represented by set T_2.

3. Exactly 3 letters put correctly represented by set T_3.
   and so on....
   But it's difficult to compute set T_i.

So it would be feasible to compute set T_i if we define the set T_i as follows:
T_i : set of combinations such that letter L_i goes to envelope E_i.
So constructing the set $\bar{S}$ as :
$\bar{S} = T_1 \bigcup T_2 \bigcup .... \bigcup T_n$
So from Principle of Inclusion and Exclusion we get:
$|T_1 \bigcup T_2 \bigcup T_3 \bigcup ...... \bigcup T_n| = \sum_i |T_i|$ - $\sum_{i<j} |T_i \bigcap T_j| + \sum_{i<j<k} |T_i \bigcap T_j \bigcap T_k|$ - .... $+(-1)^n$ $|T_1 \bigcap T_2 \bigcap T_3 \bigcap ...... \bigcap T_n|$
As, $\sum_i |T_i| = \binom{n}{1}$(n-1)!
$\sum_{i<j} |T_i \bigcap T_j| = \binom{n}{2}$(n-2)!
$\sum_{i<j<k} |T_i \bigcap T_j \bigcap T_k| = \binom{n}{3}$(n-3)!
And so on....
So, $\bar{S} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$(n-k)!
So, $|S| = |\Omega|$ - $|\bar{S}|$
$|S|$ = n! - $\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$

1.2 Counting using Recurrences

Let us visualize the n letters and n envelopes as follows:
E_1...........E_i.......E_n
L_1...........L_i.......L_n
D_n : It denotes the number of derangement of n letters.
Case 1:if L_i goes to E_1 then D_{n-2}
Case 2:if L_i does not go to E_1 then D_{n-1}
So we arrive at the following recurrence relation:
$D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$
Subtracting nD_{n-1} from both sides we get:
$D_{n-1} - nD_{n-1} = -D_{n-1} + (n-1)D_{n-2}$
$A_n := D_n - nD_{n-1}$
So, $A_n = -A_{n-1} = (-1)^2 A_{n-2} = ... = (-1)^{n-2} A_2 = (-1)^{n-2} = (-1)^n$
D_n-n$D_{n-1} = (-1)^n$
D_n = n$D_{n-1} + (-1)^n$, $n >= 1$ with $D_0 = 1$

1.3 Solving Recurrence using Generating Function

Select a generating function:

$$S(x) := \sum_{n=0}^{\infty} \frac{D_n}{n!} x^n = 1 + \sum_{n=1}^{\infty} \left(\frac{nD_{n-1}+(-1)^n}{n!}\right) x^n$$
$$= 1 + \sum_{n=1}^{\infty} \frac{D_{n-1}}{(n-1)!} x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$$

$= 1 + x\sum_{n=1}^{\infty} \frac{D_{n-1}}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$

$= x\sum_{n=0}^{\infty} \frac{D_{n-1}}{(n)!} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$

Now we know that:
$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$
We get a recurrence as follows:
So, $S(x) = xS(x) + e^{-x}$
$(1 - x)S(x) = e^{-x}$

Therefore, $S(x) = \frac{e^{-x}}{(1-x)}$
So, we need to find coefficient of x^n in $S(x)$

To think about these cases and apply generating functions to solve the problems.

1. Try to solve Lucas sequence using generating functions.
2. Find the number of shortest paths from (0,0) to (n,n) in the following figure such that one cannot cross the line passing through the diagonal?

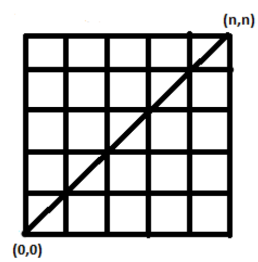

Figure 2: Fig. representing the grid of n x n.