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Lecture 17: Graph Theory

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1 Graph Theory

Definition 1 *A* ***Graph*** *is represented by* ***G(V,E)****, whereas* ***V*** *stands for set of vertices of graph and* ***E*** *stands for set of edges (*$\boldsymbol{E} \subseteq \boldsymbol{V} \times \boldsymbol{V}$*).*

- A **Simple Graph** has no multiple edges,no self loop and is undirected.
- $\forall v \in V, Ngb(v) = \{w : (v, w) \in E\}$.
- $|Ngb(v)|$ = deg(v) = d(v).

For undirected graph (v,w) and (w,v) are same.

- Degree sequence is sequence of degree of vertices,labelling the vertices gives permutation of degree sequences.
- So without loss of generality **degree sequence** of a graph is $(d_1, d_2, d_3, ..., d_n)$ whereas $(d_1 \leq d_2 \leq d_3 \leq ... \leq d_n)$

Theorem 1.1 *For every graph G(V,E)*

$$\sum d(v) = 2|E|$$

Corollary 1.2 *If* $(d_1,d_2,...,d_n)$ *is a degree sequence of a graph then,*

$$\sum_{i=1}^{n} d_i = even$$

Corollary 1.3 *Number of odd degree vertices in graph is always even.*

$$\sum_{i=1}^{n} d_i = \sum_{even} d_e(v) + \sum_{odd} d_o(v) = even.$$

- $(d_1,d_2,...,d_n)$ is a valid degree sequence of a simple undirected graph,then each of d_i's will satisfy : $0 \leq d_i \leq$ n-1.

- Let we have ordered degree sequence such that first (i-1) vertices has degree 0 and $d_i \neq 0$.

Theorem 1.4 *In every connected graph there exist atleast two vertices with same non-zero degree.*

Proof.

- $(d_1, d_2, ..., d_n)$ is a valid degree sequence of a simple undirected graph,then each of $d_i^{'s}$ will satisfy : $0 \leq d_i \leq n-1$.

- Let we have ordered degree sequence such that first (i-1) vertices has degree 0 and $d_i \neq 0$.

Let H be connected subgraph of G(V,E) with n vertices.
$\therefore$vertices iin H is n-(i-1) and every degree is $1 \leq d_i \leq (n-(i-1))-1 = n-i$
$\because$ total number of vertices in H is n-(i-1) and degree is between 1 to n-i,
$\therefore$ by pigeon hole principle(PHP),at least 2 has same degree.

□

Theorem 1.5 $\Delta G \leq$ *total number of vertices with non zero degree.*
Proof. by pigeon hole principle(PHP) as described above. □

(ΔG is defined as maximum degree of a vertex in graph).

2 Graph Isomorphism

Given two different graph G_1 and G_2 such that,
$G_1(V_1, E_1)$
$G_2(V_2, E_2)$
these two graph are **isomorphic** if their is a function 'p' on V_1 and V_2
p : $V_1 \longrightarrow V_2$ (bijection)
such that,(v, w) $\epsilon E_1 \Leftrightarrow (p(v), p(w)) \epsilon E_2$

If degree sequence of $G_1 \neq$degree sequence of G_2 then G_1 is **not isomorphic** to G_2.

3 Tree

Definition 2 *A maximum connected acyclic graph is tree.*
Number of edges is n-1 with total number of vertices is n.

The following are equivallent

- *A tree is minimally edge connected graph.(removing any single edge will disconnect graph).*

- *A tree is maximally acyclic graph.(including any single edge will create a cycle).*
- *A tree is a connected graph with n-1 edges.*
- $\forall$ *u,v,*$\exists$ *a unique path from u to v*

Lemma 3.1 *If G is a tree then their exist two vertices with degree 1.if not then their exist a cycle and it is not a tree.*

4 Connected Component

If G = $H_1 \bigcup H_2 \bigcup ... \bigcup H_n$ whereas,
$H_1(V_1, E_1)$,$H_1(V_2, E_2)$,...,$H_n(V_n, E_n)$ such that,
$V_1, V_2, ..., V_n$ is partition of V.
$E_1, E_2, ..., E_n$ is partition of E.

- H_i is connected.
- $\forall$ (u,v) , $u \epsilon V_i$,v ϵV_j (j $\neq$ i) then (u,v) $\notin$ E.

4.1 k-connected

Graph is **2-connected** if $\forall v$,$G \setminus v$ is connected.
Graph is **2-edge-connected** if $\forall e$,$G \setminus e$ is connected.
Graph is **k-connected** if $\forall (v_1, v_2, ..., v_{k-1})$,$G \setminus (v_1, v_2, ..., v_{k-1})$ is connected.

5 Bipartite graph

Definition 3 *A graph G(V,E) in which V can be partitioned into two subsets such that their is no edge between vertices of any one set.*

Property :

- Their exists no cycle with odd number of vertices.
- Partition V = $V_1 \bigcup V_2$,s.t. E $\subseteq$V$_1 \times V_2$