Discrete Mathematics	10/10/2018

Lecture 19: Spanning tree and Matching

Instructor: Sourav Chakraborty *Scribe: Shahansha Salim*

1 Spanning Tree

Definition 1 ***Spanning tree*** *T of an undirected graph G(V,E) is a subgraph that is a tree with all vertices of G and is minimally connected. T = (V,E'). It has no cycle.*

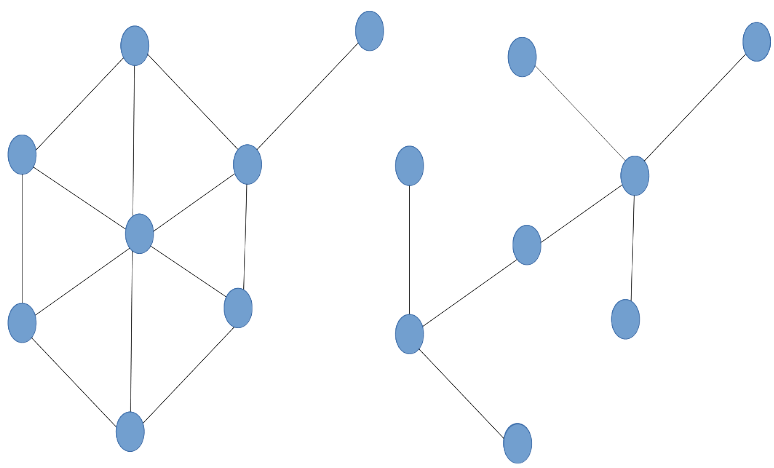

Figure 1: Graph G and its spanning tree

Theorem 1.1 *Every connected graph has atleast one spanning tree.*

Proof. Let G be a connected graph. If G has no cycles, then it is its own spanning tree.If G has cycles, then on deleting one edge from each of the cycles, the graph remains connected and cycle free containing all the vertices of G. □

Algorithm to find spanning tree is also same as this. Add the edges one by one and if it forms a cycle, then discard the edge. Spanning trees can be thought as the skeleton of a graph. There can be more than one spanning tree for a graph.

2 Paths

Definition 2 ***Hamiltonian Path*** *is a path in an undirected or directed graph that visits each vertex exactly once.*

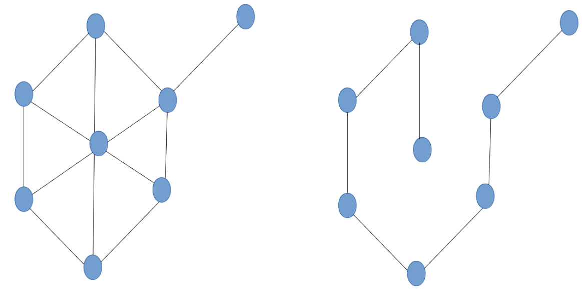

Figure 2: Graph G and its hamiltonian path

Tournament always have a directed hamiltonian path.
Graph G given below has no hamiltonian path.

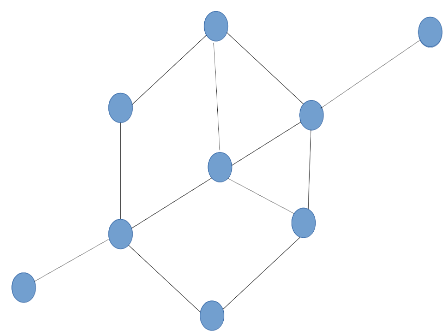

Figure 3: Graph G which does not have hamiltonian path

Definition 3 ***Hamiltonian Cycle*** *is a cycle in an undirected or directed graph that visits each vertex exactly once*

Figure 4 shows a graph G and its hamiltonian cycle.

Fact 1 *Finding Hamiltonian path and Hamiltonian cycle in undirected graph is np-complete*

Definition 4 ***Eulerian Path*** *is a path that touches every edge exactly once.*

A graph has no Eulerian path if it has more than two odd degree vertices.

Definition 5 ***Eulerian Cycle*** *is a Eulerian path that starts and ends on same vertex*

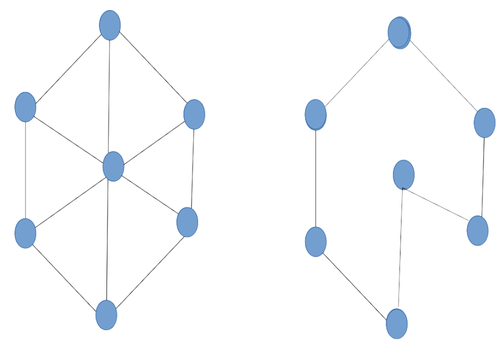

Figure 4: Graph G and its hamiltonian cycle

Theorem 2.1 *Eulerian cycle exists iff all the degrees of vertices are even*

Proof. Let G(V,E) be a connected graph. If all the vertices have even degree, we can write it as a union of cycles(this has been proved already previously). to prove $\Longleftarrow$ direction, Let Z_1 be one of the cycles of this partition. If G consists only of this cycle, then G is obviously eulerian. Otherwise, there is another cycle Z_2 with a point v in common with Z_1. The walk beginning at v and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the lines of these two cycles. By continuing this process, we can construct a closed trail containing all the lines of G. Hence G is eulerian. □

Definition 6 ***Cut Edge or Bridge*** *is an edge of a graph whose deletion increases the number of connected components.*

3 Matching

Definition 7 ***Matching*** *M in G is a set of pairwise non-adjacent edges,none of which are loops; that is, no two edges share a common vertex.*
matching is a graph with maximum degree 1.

A **Maximal matching** is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching, that is, M is maximal if it is not a subset of any other matching in graph G
A typical real world problem that is an application of finding a Maximum Matching is kidney donors to receivers graph. It can be modelled as a Bipartite graph in which we have to find a maximum matching.

A **Maximum matching** is a matching that contains the largest possible number of edges

Augmented Paths: Take a run of Matched/Unmatched paths and augment them to find bigger Matchings.
Alternating path: A path that alternates between Matched and Unmatched edges

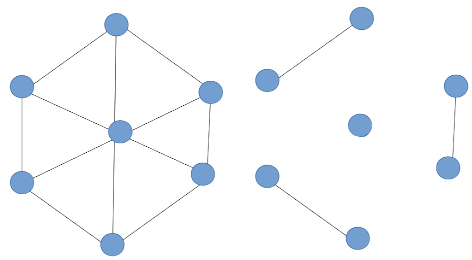

Figure 5: Graph G and its matching

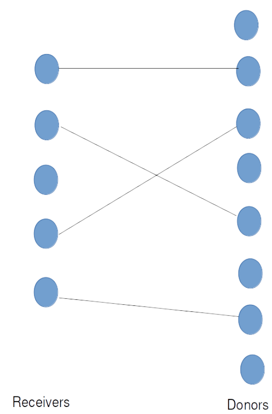

Figure 6: Kidney donors and receivers being matched

Definition 8 ***Perfect Matching*** *is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.*

Problem 1 *When does a bipartite graph G = (L,R,E) have a perfect matching on L?*

G has a perfect matching on L $\iff$ $\forall$ S $\subseteq$ L and if we take the neighbors of S denoted by N(S),
$|S| \leq |N(S)|$. This is called Hall's marriage theorem.

Definition 9 ***Vertex Cover*** *VC(G) is a set* $S \subseteq V$ *such that* $\forall (u,v) \in E$ *, either* $u \in S$ *or* $v \in S$.

A **Minimum vertex cover** is a vertex cover having the smallest possible number of vertices for a given graph.

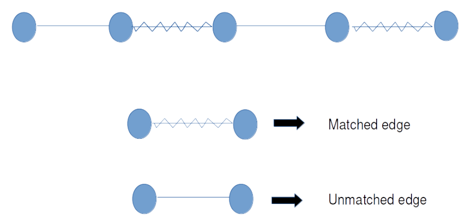

Figure 7: Alternating path

Problem 2 *What is the relation between minimum Vertex Cover C and Maximum matching M on a graph G?*

1. **C $\geq$ M**

any vertex cover should have at least one edge from the maximum matching M. in case of perfect matching, C = M. this is true for all graphs.

2. $\forall$ **G, 2M $\geq$ C $\geq$ M**

if we take both the end points for all edges in M, there will be 2M vertices. Every edge either Matched or Unmatched should have at least one vertex in the matched vertices.

Fact 2 *finding vertex cover is a NP-Complete problem*

Fact 3 *Finding matching is a polynomial time algorithm.*

Fact 4 *Finding vertex cover in a bipartite graph is not NP-Complete problem.*

Theorem 3.1 ***Konig's theorem***

If G is bipartite , then Minimum Vetrtex Cover(C) = Maximum Matching(M)

Proof.

Let M be a maximum matching in G.
Construct a Vertex Cover S. if (u,v) is a matched edge, then v $\in$ S if $\exists$an alternating path starting from an unmatched vertex ending at v.
If u and v does not have an alternating path starting at an unmatched vertex, then choose u into S.

Claim

The set S is a vertex cover.

Proof by contradiction

Let S be not a Vertex Cover. i.e., $\exists$(u,v)$\in$E that is not covered.
If (u,v) is not covered, then either (u,v) is not in the matching, or atlest one of u or v is

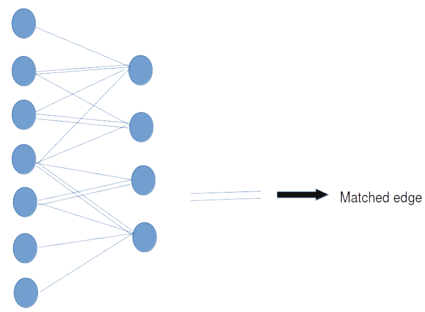

Figure 8:

matched.
Case 1.
u is unmatched and v is matched. $\implies$ v $\in$ S
$\implies$ $\impliedby$
Case 2.
u is matched and v is unmatched

u is matched $\implies$ (u,u') is in M(see figure 9)
$\implies$ u' is in S
$\implies$ u' is in R
$\implies$ there must have been an alternating path P ending at u' from unmatched edge at L. then P-u-v is an augmenting path. this is not possible as per maximality of M.
$\implies$ $\impliedby$

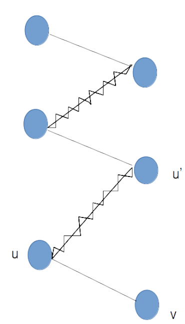

Figure 9:

Case 3.

Both u and v are unmatched.
(u,u') $\in$ M
$\implies$ $\exists$an alternating path P to u'
Consider P-u-v. it is an alternating path to v.
by definition, v would have been in S. $\implies \impliedby$ $\square$