Discrete Mathematics 24-Oct-2018

Lecture 21: Planar Graph

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1 Planarity

1.1 Observations

Observation 1.1 *K_5 or $K_{3,3}$ cannot be drawn on a plane such that there is no intersection of two edges.*

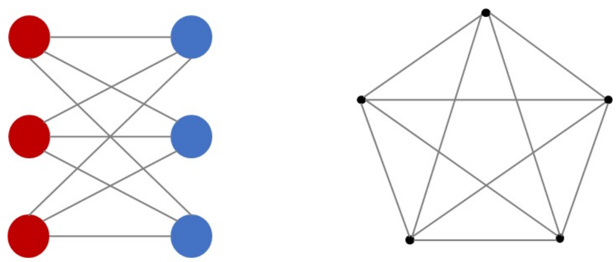

Figure 1: $K_{3,3}$ and K_5

Lemma 1.2 *If G can be drawn(no two edges intersect) on R^2 $\implies$ G can be drawn on a sphere.*

Proof :

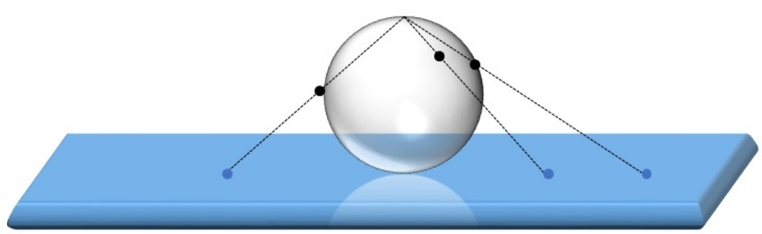

Figure 2: every point on the sphere has a bijection to a point on the plane

As every point on the sphere has bi-jection to a point on the plane.
• *The number of points on the sphere is one more than the number of points on the plane.*

1.2 Operations

- If a graph is planar , then if some vertices or edges be removed then the resulting graph would also be planar.
- If a graph is planar , then if some adjacent vertices be collapsed then the resulting graph would also be planar.

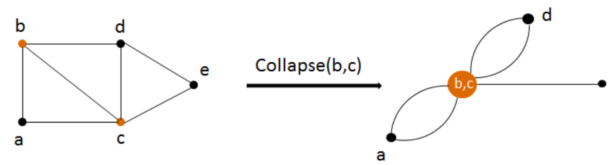

Figure 3: A planar graph after collapsing two vertices remains planar

How to check if a graph is planar?

1. Delete a vertex
2. Delete an edge
3. Contract an edge(Collapse two adjacent vertices).

   if ∃ a sequence of moves 1,2,3 such that G → H then we say G has a minor H.

   If a graph G has K_5 or $K_{3,3}$ as minor then G is not planar.

1.3 Theorems

Theorem 1.3 *A graph G is planar <=> G has no K_5 or $K_{3,3}$ as minor. (Kuratowski's theorem).*

Proof. Next lecture □

Theorem 1.4 *If G is a connected planar graph then $|V|+|E|-|F| = 2$, where $|V|$: number of vertices, $|E|$: number of edges, $|F|$: number of faces of the graph.*

Proof.

Induction on number of edges of the graph, i.e. $|E|$.
Base Case : When $|E| = 1$, as the graph is connected, so when there is a single edge means there are only 2 vertices.
$|V| = 2, |E| = 1, |F| = 1$.
so, $|V| + |E| - |F| = 2 + 1 - 1 = 2(proved)$.

Induction Hypothesis : For a planar connected graph with number of edges $\leq k$ the given statement is true.

Inductive step : Let G be a planar connected graph with number of edges = k+1. Every edge in a cycle is a part of 2 faces.
$\exists$ a cycle $\implies$ $\exists$ edge e, such that e is a part of 2 faces.
Let G' be a sub-graph of G such that $G' = G - e$.
G' is connected as e is a part of a cycle and G' is also planar.
Then,$(|V(G')| = |V|, |E(G')| = |E| - 1, |F(G')| = |F| - 1$.
According to I.H. $|V(G')| - |E(G')| + |F(G')| = 2$
$|V| - (|E| - 1) + (|F| - 1) = 2$
$|V| + |E| - |F| = 2$(Proved).

□

1.4 Definitions

Definition 1.5 *Maximal planar graph : It is a planar graph and if any more edge is introduced then it will not be planar anymore.*

1.5 Lemmas

Lemma 1.6 *For a planar graph,* $|E| \leq 3|V| - 6$.

Proof : Every edge in a maximal planar graph is a part of a cycle and as a result part of 2 faces. So $\sum_i |E(f_i)| = 2|E|$.

Now as every face contains at least 3 edges, so $3|F| \leq \sum_i |E(f_i)|$.

Equating the above two equations, $3|F| \leq 2|E|$.

so,$|F| \leq \frac{2}{3}|E|$.

From Euler's formula, $2 = |V| - |E| + |F|$
$2 \leq |V| - |E| + \frac{2}{3}|E|$
$2 \leq |V| - \frac{1}{3}|E|$
$|V| - 2 \geq \frac{|E|}{3}$
$|E| \leq 3|V| - 6$

Lemma 1.7 *For a bipartite planar graph,* $|E| \leq 2|V| - 4$.

Proof : As there is no odd cycle in a bipartite graph so every edge in a bipartite planar graph is a part of a 4-cycle and part of 2 faces. So $\sum_i |E(f_i)| = 2|E|$.

Now as every face contains at least 4 edges, so $4|F| \leq \sum_i |E(f_i)|$.

Equating the above two equations, $4|F| \leq 2|E|$.

so,$|F| \leq \frac{1}{2}|E|$

From Euler's formula, $2 = |V| - |E| + |F|$
$2 \leq |V| - |E| + \frac{1}{2}|E|$
$2 \leq |V| - \frac{1}{2}|E|$
$|V| - 2 \geq \frac{|E|}{2}$
$|E| \leq 2|V| - 4$