Discrete Mathematics	26 Oct 2018

Lecture 22: Proof Of Kuratowski's theorem

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1 Preliminaries

1.1 Definitions

1. **Plannar Graph** : A planar graph is a graph that can be embedded in the Euclidean plane, i.e., it can be drawn on the plane in such a way that no edges cross each other. Such a drawing is called a **plane graph** .

2. K_5 is a non-plannar graph with minimum number of edges on 5 vertices and $K_{3,3}$ is a non-plannar graph with minimum number of edges on 6 vertices.

3. A **minimal non-planar** graph is a non-planar graph G such that every proper subgraph of G is planar.

4. **Minor of a Graph** : An undirected graph H is a minor of another undirected graph G if H can be obtained from G by
   1. contracting some edges
   2. Deleting some edges
   3. Deleting some vertices.

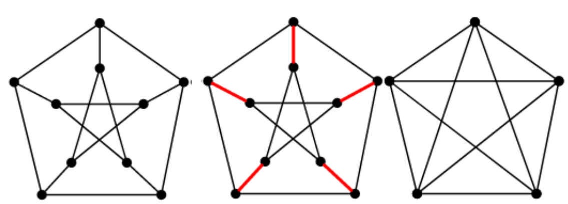

Figure 1: Petersen graph has K_5 as minor. Red edges are contracted to get K_5.

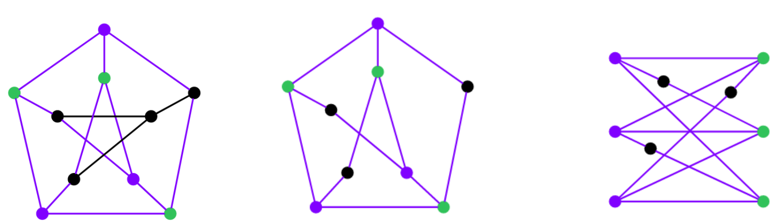

Figure 2: Petersen graph has $K_{3,3}$ as minor. Black edges are deleted to get $K_{3,3}$.

1.2 Lemmas

1. **Lemma 1** For every face of a given plane graph G, there is a drawing of G for which the face is exterior.

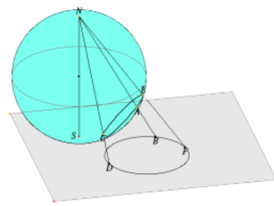

Figure 3: We draw the graph on a sphere, and then project it from a point on the face f

2. **Lemma 2** Every minimal non-planar graph is 2-connected.

   proof Assume, for contradiction, that there exists a minimal non-planar graph G =(V, E) that is not 2-connected.
   There exists a vertex v whose removal disconnects G. Let C1,C2 be the components of G/v. By the minimality of G, the induced subgraph on C1 $\cup$ {v} and C2 $\cup$ {v} are both planar. We can embed both graphs with v on the unbounded face, and merge both copies of v to get a planar graph. This is contradiction.

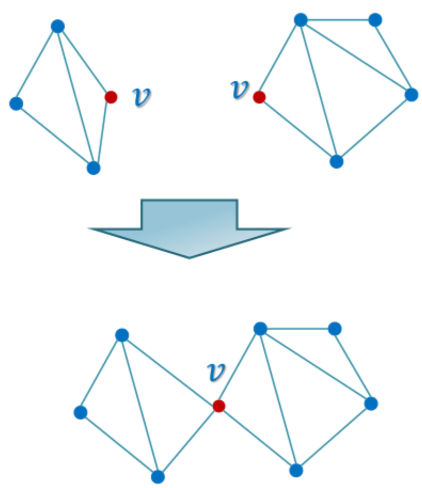

Figure 4: resulting graph is again planar

□

3. **Lemma 3** Let G = (V, E) be a graph with fewest edges among all non-planar graphs

without a K_5 or $K_{3,3}$ as minor.Then G is 3-connected.

proof Assume, for contradiction, that there exists a minimal non-planar graph G =(V, E) without a K_5 or $K_{3,3}$ as minor is not 3-connected. There exists a pair of vertices u,v whose removal disconnects G.Let C1 , C2 be the components of G / v. By the minimality of G, the induced subgraph on C1 ∪ {u,v} and C2 ∪ {u,v} are both planar. Now there arises two cases:

(a) **Case 1** : Edge (u,v) is present in the graph G. Then C1 ∪ {u,v} and C2∪ {u,v} both are planar, by minimality of G. Now we can bring the edge {u,v} on the outer face of C1∪{u,v} and C2∪{u,v} using lemma 1. Since both graphs are planar, by joining vertices u in C1 ∪ {u,v} to u in C2 ∪ {u,v}and vertices v in C1∪ {u,v} to v in C2∪ {u,v} craetes a planar graph. Contradiction to our assumption that G is non planar.

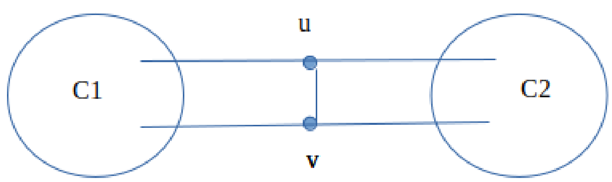

Figure 5: Graph G which is minimal non-planar graph

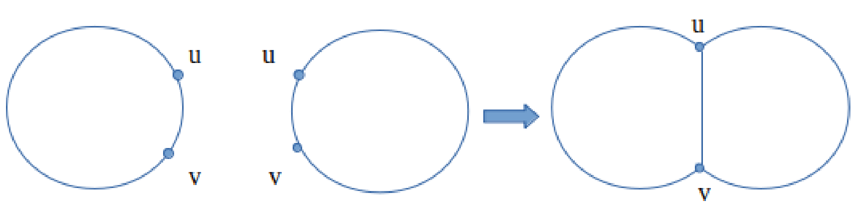

Figure 6: resulting graph is again planar

(b) A **Case 2** : Edge {u,v} is not present. Then C1 ∪ {u,v} and C2 ∪ {u,v} both are planar, by minimality of G. Since u,v are not connected by direct edge, by bringing the vertex u on the outer face of C1 ∪ {u,v} might not bring the v on outer face of C1 ∪ {u,v} simultaneously and vice versa. So the trick we apply is to join u and v by an edge temporarily and then we can bring both the vertices u,v on the outer face.

Lets call G1 = C1 ∪ {u,v} ∪ (u,v) and G2 = C2 ∪ {u,v} ∪ (u,v). Here we cannot guarentee that G1 and G2 are planar. Because we have added an extra edge (u,v). But are we doing any mistake by adding extra edge? Does this introduce K_5 or $K_{3,3}$ in the graph C1 ∪ {u,v} or C2 ∪ {u,v}? The answer is No. Because if it would have introduced a K_5 or $K_{3,3}$ in the graph C1 ∪ {u,v} or C2 ∪ {u,v} that means the path joining u and v through graph C2 ∪ {u,v} contains a K_5 or $K_{3,3}$.This implies the original graph G contains a K_5 or $K_{3,3}$ as minor. Which is contradiction , because we startd with a non planar graph G which doesn't contain K_5 or $K_{3,3}$ as minor.

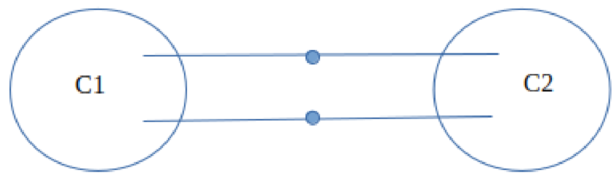

Figure 7: resulting graph is again planar

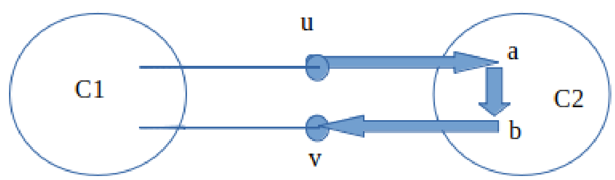

Figure 8: We can contract the path u→a→b→v to make an edge. And if C1 ∪ {u,a,b,v} contains K_5 or $K_{3,3}$ then G has K_5 or $K_{3,3}$.

□

1.3 Proof

Theorem 1.1 (Kuratowski's Theorem) *A graph is planar iff it does not have K_5 or $K_{3,3}$ as minors.*

proof We know that if a graph contains K_5 or $K_3, 3$ as a minor graph, then it is not planar.
It remains to prove that every non-planar graph contains K_5 or $K_3, 3$ as minor.
Proof Strategy: For proving this

1. It suffices to prove this only for minimal non- planar graphs.

2. We will show that every minimal non-planar graph with no K_5 or $K_{3,3}$ as minor must be 3-connected.

3. We then show that every 3-connected graph with no K_5 or $K_{3,3}$ as minor is planar. But we started with a non-planar graph. which is Contradiction! So a non-planar graph must contain K_5 or $K_{3,3}$ as minor graph.

We need to show that if a graph is non-planar then it must contain a K_5 or $K_{3,3}$ as minor graphs. Let G be the smallest graph in the set of non planar graph which is 3-connected. Then we prove the contra-positive of statement "G doesn't have K_5 and $K_{3,3}$ as minor and G is non-planar" i.e. "G is planar or G has K_5 or $K_{3,3}$ as minor".

1. **Case 1** : Let denote N(v) as set of all vertices neighbour to v. Then if |N(u)∩N(v)| >=3 then G has a K_5 a minor. Which is contradiction.

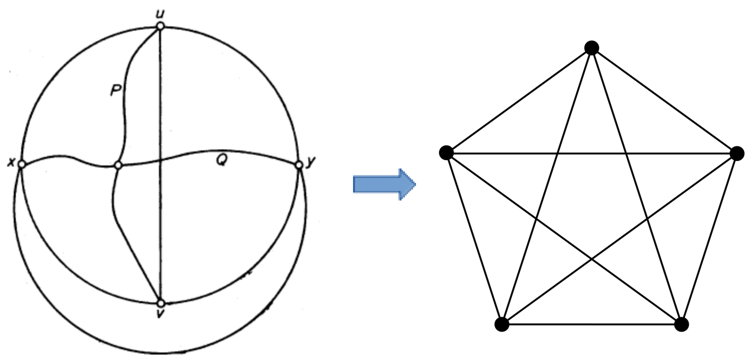

Figure 9: This graph has K_5 as minor.

2. **Case 2** : Here |N(u)∩N(v)| <= 2 is satisfied but it is also non planar. Let
N(u)=a_1,a_2,a_3,....,a_n
N(v)=b_1,b_2,b_3,....,b_n
If the neighbour of u and v interleave then G has a $K_{3,3}$ minor. Which is contradiction.

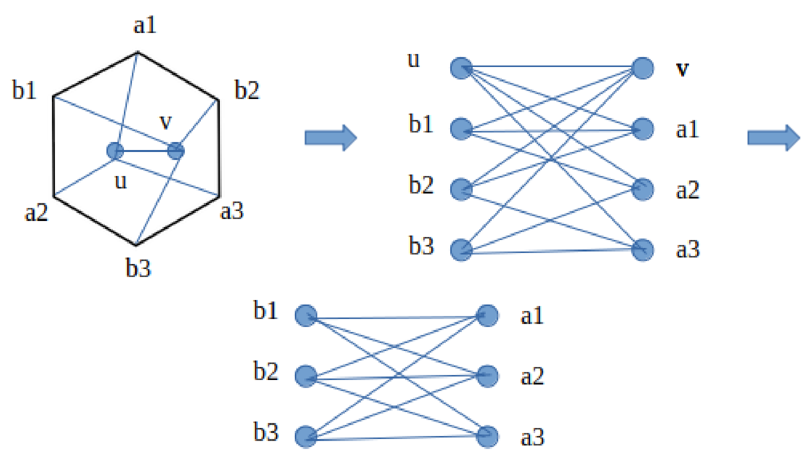

Figure 10: This graph has $K_{3,3}$ as minor.

3. **Case 3** : If the neighbour of u and v doesn't interleave and |N(u)∩N(v)| <= 2 is satisfied then G is planar.

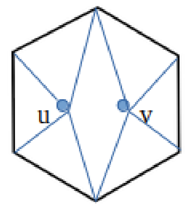

Figure 11: This is a planar graph

All the cases gives contradiction. This proves the theorem. □

1.4 References

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3. https://www.math.hmc.edu/ kindred/cuc-only/math104/lectures/lect17-slides-handout.pdf
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