Discrete Mathematics	02/11/2018

Lecture 23: Coloring

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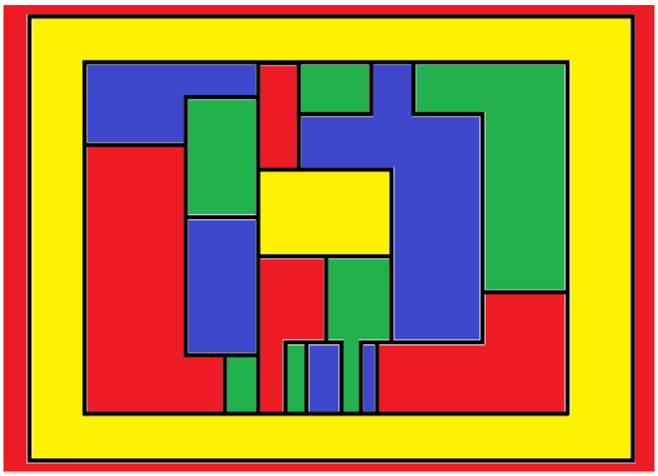
1 Map



Each region represents a Country .

Different countries have different colors associated with it.

How many Colors are sufficient to color all the countries ,s.t no 2 adjacent countries have the same color ?

2 4-Color Theorem

There exists a map ,such that 4 colors are necessary to color each Country separately.

Any Map can be Colored with 4 colors , it's one of the biggest theorem in the last 50 years .

Any Planar graph can be colored with 4 Colors.

*Proof of the 4 Color theorem is not possible and is still an unsolved problem.

3 5-Color theorem

Every Planar graph can be colored with 5 colors.
Properties of Planar graph

- $n - m + r = 2$, Euler's formula for planar graph

- $m \leqslant 3n - 6$, maximal planar graph condition
- Bipartite
  $m \leqslant 2n - 4$
- K_5 and $K_{3,3}$ minors and subdivisions must not be present.

Q.What is the upper bound on the degree of a planar graph ?

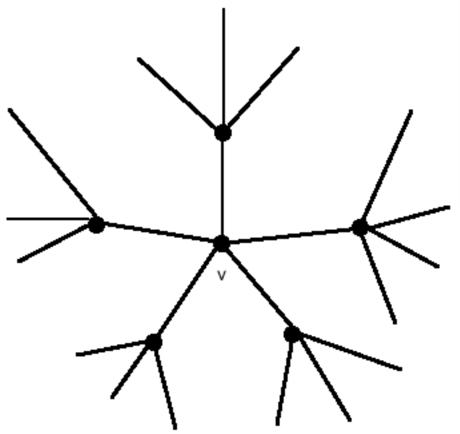
$\rightarrow$It can be anything,but the minimum degree is $\leqslant 5$

Proof: Suppose in a planar graph G , the degrees are $d_1, d_2, d_3, ..., d_n$.
$\sum_{i=1}^{n} d_n = 2$m , where m=number of edges in G
again from Eulers Formula we have ,
$n - m + r = 2$,using this we have $d_i \leqslant 5$.

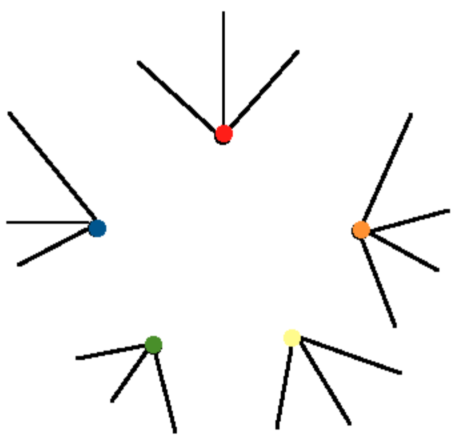
Proof of 5 Color Theorem :

To Prove:For a connected planar simple graph G, the vertices in G can be coloured with 5 or fewer colours for a good colouring of G. S(k)

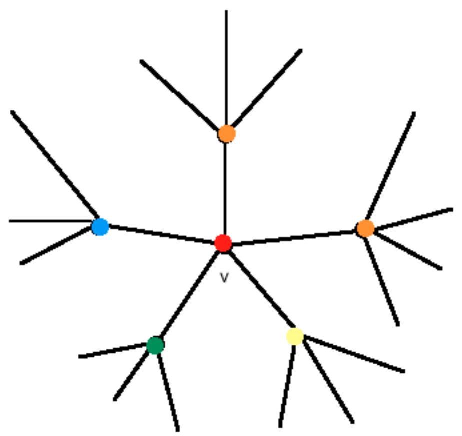
- Base Step: For, $1 \leqslant n \leqslant 5$, this is trivially true. A graph on 1 vertex can easily be coloured with just 1 colour, while a graph with 5 vertices can easily be coloured with just 5 colours for a good colouring.
- Induction Step: Suppose that for all $k \geqslant 2$, S$(k-1)$ is true. That is, for all connected planar simple graphs on $k-1$ vertices, we can obtain a good colouring of the vertices in G with 5 or fewer colours. We want to verify that S(k) is true (that for all connected planar simple graphs on k vertices, we can obtain a good colour of the vertices in G with 5 or fewer colours still).
- Once again, suppose we have a graph G on k vertices. We know that a connected planar simple graph G contains a vertex of degree 5 or less.



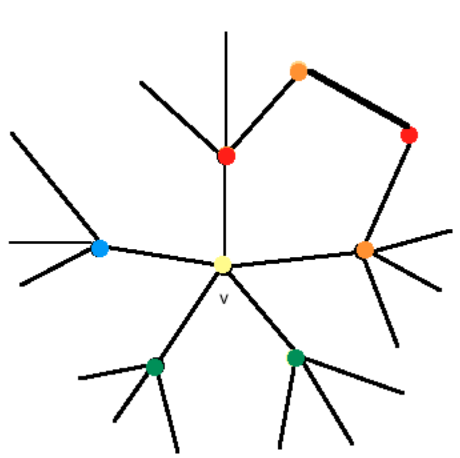
- Suppose the v has deg(v)=5. If we delete this vertex and all edges incident with v, then by our induction hypothesis the resulting graph has a good 5 colouring.



- Now we reinsert the vertex v. Notice that a good 5-colouring cannot happen if vertex v has neighbours all with different vertex colours since v would then need a 6^{th} colour. We will prove that the neighbours of v cannot all be the same colour now with the following two cases. We will arbitrarily select the red and orange vertices for these cases without loss of generality.
- CASE 1: If there no is red-orange alternating vertex path starting at the red vertex and ending at the orange vertex, then we can interchange the red vertex with an orange vertex and our proof is complete:



- CASE 2: If there is a red-orange alternating vertex path starting at the red vertex and ending at the orange vertex, then inspect the yellow and the green vertices. There cannot be an alternating yellow-green vertex path starting at yellow and ending at green since the red-orange path interrupts this path. Hence the yellow vertex can be interchanged green and the proof is once again complete.



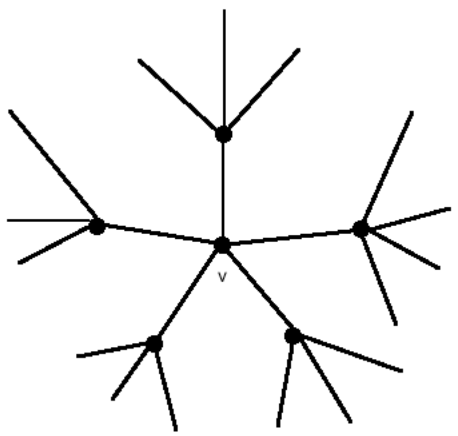
- Hence $S(k-1)$ implies S(k). By the principle of mathematical induction, for $n \geqslant 1$, S(n) is true

4 6 Color Theorem

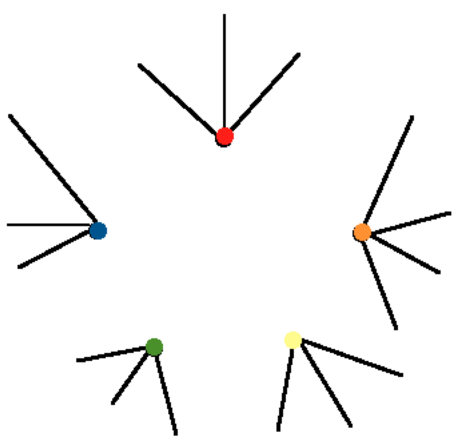
To Prove : For a connected planar simple graph G, the vertices in G can be coloured with 6 or fewer colours for a good colouring of G. S(k)

- Base Step: For $1 \leqslant n \leqslant 6$, this is trivially true. A graph on 1 vertex can easily be coloured with just 1 colour, while a graph with 6 vertices can easily be coloured with just 6 colours for a good colouring .

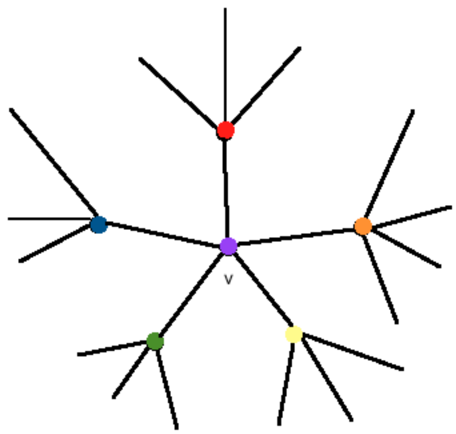
- Induction Step: Suppose that for all $k \geqslant 2$, $S(k-1)$ is true. That is, for all connected planar simple graphs on $k-1$ vertices, we can obtain a good colouring of the vertices in G with 6 or fewer colours. We want to verify that S(k) is true (that for all connected planar simple graphs on k vertices, we can obtain a good colour of the vertices in G with 6 or fewer colours still).
- Now let G be a connected planar simple graph on k vertices. Recall that a connected planar simple graph G contains a vertex of degree 5 or less. Suppose the vertex v has deg(v)=5



- Now suppose that we remove vertex v and all of the edges incident with v. This graph now has less than k vertices, and by our induction hypothesis, we know this resulting graph can be coloured with 6 or fewer colours



- Adding vertex v back, we know that the neighbourhood of v contains 5 members. Hence if we use the 6th colour for vertex v, our proof is complete.



- Hence $S(k-1)$ implies S(k). By the principle of mathematical induction, for $n \geqslant 1$, S(n) is true.

References

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- Graph Theory by D.B.West
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