Discrete Mathematics 2-11-2018

Lecture 24: Flow-Cut and Menger's Theorem

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1 Flow and Cut

Let there be a directed graph G=(V,E)
$c : E \Rightarrow \mathbb{R}$
$s, t \in V$
$f : E \Rightarrow \mathbb{R}$ such that

1. $\forall v \neq s, t$

$$\sum_u [f(u, v)] = \sum_u [f(v, u)] \tag{1}$$

2. $f(e) \leq c(e)$

$$value(f) = \sum_u [f(s, u)] - \sum_u [f(u, s)] = \sum_u [f(u, t)] - \sum_u [f(t, u)]$$

$$V = A \cup B \tag{2}$$

$$C = \sum_{(u,v) \in E u \in A, v \in B} [c(u, v)] \tag{3}$$

1.1 Proofs

Theorem 1.1 *Max s.t flow $\leq$ Min s.t cut(sum of the capacity of edges)*

Proof. As we can't send more than the capacity of a Bridge, hence maximum s.t. flow is always less than or equal to the capacity of the Bridge. □

Theorem 1.2 *Max s.t flow = min s.t cut*

Proof. **Algorithmic Approach :-**

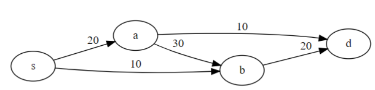

Figure 1:

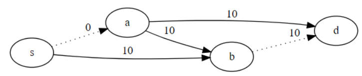

Figure 2:

Residual graph = Graph with capacity and flow such that
back edge = flow and forward edge =capacity-flow
As long as there is a path from s to t in the residual path, we can increase the net flow. So keep constructing the residual graph until there is no s-t path in the graph.

Lemma 1.3 *If the total flow $<$ min cut then $\exists$ a s.t. path in the graph.*

Hence, maximum flow in a path = minimum of the capacity □

Theorem 1.4 *[Hall's Marriage Theorem] $\nexists$ a perfect matching between A and B iff $\forall$ S $\subseteq$ A such that $|ngb(S)| \geq |S|$*

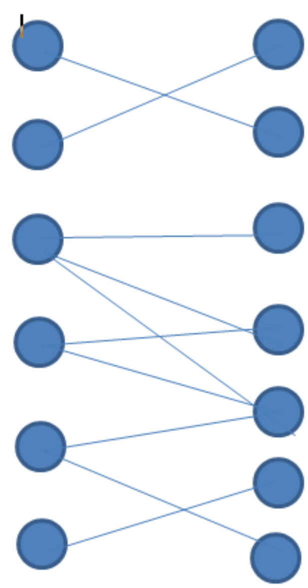

Figure 3:

Proof. **Contrapositive :-**
If $\nexists$ a perfect matching, then $\exists$ S $\subseteq$ A such that $|ngb(S)| < |S|$
|A|=$|B|$=K It is trivially true that if max flow =K, then $\exists$ perfect matching. Hence, it is only left to prove the iff $\exists$ perfect matching, then max flow= K.(OR)
If $\nexists$ max flow of size K $\Rightarrow \nexists$ a cut of size $<$ K.

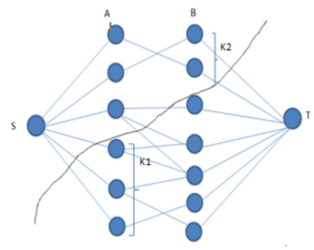

Figure 4:

Let K1 be the no. of vertices in A that is in cut in s-side.
Let K2 be the no. of vertices in B that is in cut in t-side.
Let r be the no. of vertices in A and B that is in cut in A-B side.
So, K1+K2+r<K
$|ngb(S)| \leq K2 + r < K - K1 < |S|$ □

Theorem 1.5 *[Menger's Theorem] If G is directed K-edge connected then, ∀ u,v ∃ K edge disjoint path from u to v.*

Proof. Suppose K=2.

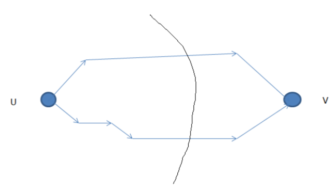

Figure 5:

Atleast two edges going at right direction should be present in the cut or otherwise it won't be 2-edge connected. min u-v cut ≥ 2 ∃ integer flow of capacity ≥ 2 Remove a u-v path from the flow graph, then the remaining flow graph has a flow of size ≥ 1. By same argument, we can extend it to K-edge connected.

□