Discrete Mathematics 20/7/2018

Lecture 1: Relation and Function

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1 Relation

1.1 Examples of Relations

1. Let us take $\mathbb{N}$ to be the set of natural numbers and take the binary relation '$\leq$'. For any $a \in \mathbb{N}$, $a \leq a$. So this relation is **Reflexive**. Now for any $a, b, c \in \mathbb{N}$ if $a \leq b$ and $b \leq c$ then it's obvious that $a \leq c$, which implies that the relation is **Transitive**. Now $2 \leq 3$ but $3 \nleq 2$, so it's not symmetric. If $a \leq b$ and $b \leq a$ then $a = b$, which implies that '$\leq$' is **Anti-symmetric**. Now this relation is also **Connex** because for any $a, b \in \mathbb{N}$ either $a \leq b$ or $b \leq a$ or both.

2. Suppose $S = \{1, 2, 3\}$ and take power set of S i.e. $P(S)$ and on $P(S)$ take the relation 'subset of' i.e '$\subseteq$'. Here $\{1\}$ is not related to $\{2\}$ because $\{1\} \nsubseteq \{2\}$ and also $\{2\} \nsubseteq \{1\}$. So this example showing that not every relation is Connex.

As we can think a relation as a graph then any relation which is Symmetric and Connex can be reprented as a Complete graph. And if a relation is Anti-symmetric and and Connex wil be a Tournament.

1.2 Equivalence Relation

Definition 1 *A relation is said to be **Equivalence Relation** if it is **Refelexive, Symmetric** and **Transitive**.*

As for an example let us take the binary relation of equality,'=' on the set $\mathbb{N}$. Then this '=' is an equivalence relation on $\mathbb{N}$ since for $x, y, z \in \mathbb{N}$,

- (Reflexive): $x = x$.
- (Symmetric): $x = y$ the $y = x$.
- (Transitive): $x = y$ and $y = z$ then $x = z$.

Definition 2 (Partition of a set) *A partition of a set X is a set of nonempty subsets of X such that every element x in X is in exactly one of these subsets (i.e., X is a disjoint union of the subsets).*

Equivalently,a family of sets P is a partition of X if and only if all of the following conditions hold:

- *The family P does not contain the empty set.*
- *The union of the sets in P is equal to X.(i.e. $\cup_{A \in P} = X$)*

- *The intersection of any two distinct sets in P is empty(i.e. for all $A, B \in X$, $A \cap B = \emptyset$). So the elements of P are pairwise disjoint.*

Example of Partition of a set: Let S={1,2,3} then {1,2} and {3} is a partition of the set S. Similarly {1},{2} and {3} is also an another partition of S. Here the ordering of the partitions do not matter unless it is declared.

Problem 1 *How many ways we can partition a set of size n into k parts such that,*

1. *the ordering among the parts does not matter.*
2. *the parts have an ordering that means the ordering among the parts matter.*

Definition 3 (Equivalence Class) *Let 'R' be an equivalence relation on a set X. Then for $a, b \in$X if aRb we say that a and b are in the same equivalence class of 'a'. We denote the equivalence class of 'a' as $[a]$.*

Theorem 1.1 *Let R be an equivalence relation on the set X and T($\subseteq$X) is an equivalence class. Now prove that for all b,c $\in$ T, bRc.*

Proof. Consider $T = [a]$, for some $a \in X$. Now from $b, c \in [a]$ we know that aRb and aRc. By using symmetry of R we get bRa and cRa. Now bRa and aRc implies bRc (using Transitivity) and this completes the proof. □

Let us assume that R is an equivalence relation on X.

Fact 1 *If $b \in [a]$ and $bRc \implies c \in [a]$. Because $b \in [a] \implies aRb$ and also bRc (given). So, aRb and bRc both implies aRc i.e. $c \in [a]$.*

Fact 2 *If $b \in [a]$ then $[b] = [a]$. The proof of this fact is left as an exercise.*

Fact 3 *For all $a, b \in X$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.Proof of this is left an exercise.*

Fact 4 *X is the union of all the equivalence classes with respect to R i.e. $X = \cup_{a \in X}[a]$. Proof of this left as an exercise.*

Theorem 1.2 *Let R be an equivalence relation on a set X. Then equivalence classes forms a partition of X.*

Proof. Proof of the above facts concludes this theorem. □

Note: Equivalence relation decomposed an undirected graph into complete graphs.

Definition 4 *Consider an undirected graph $G = (V, E)$. For any $u, v \in V$ we say **u 'is connected to' v** if there exist a sequence of vertices $v_1, v_2, ..., v_k$ such that $(u, v_1), (v_1, v_2), ..., (v_k, v) \in E$ i.e. for all $1 \leq i \leq (k-1)$, $(v_i, v_{(i+1)}) \in E$ and $(u, v_1), (v_k, v) \in E$.*

Problem 2 *Prove that the 'is connected to' relation is an equivalence relation. What are the equivalence classes of this relation?*

1.3 Ordering

Definition 5 (Order/Partial Order) *Let '$\sqsubseteq$' is a relation on a set, S (i.e. '$\sqsubseteq$' $\subseteq S \times S$). '$\sqsubseteq$' is an order on S if for any $a, b \in S$, $a \sqsubseteq b \implies b \sqsubseteq a$ unless $a = b$.*

So it's easy to check that this relation is Anti-symmetric, Reflexive and Transitive. So we can also say that if a relation is Anti-symmetric, Reflexive and Transitive then it is 'partial order' or simply 'order'. A set with a partial order is said to be partial ordered set or Poset. We can represent order relation as a graph but that can be chaotic. So there is another representation of order relation that is going to be discussed in the following.

Poset Diagrm(Hasse Diagram): Let $\sqsubseteq$ be a partial order relation on the set S. The Poset diagram is as follows:

- There is a vertex corresponding to each element of 'S'.
- A directed edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in S$ such that $a \sqsubseteq x$ and $x \sqsubseteq b$.
- A directed edge between the elements 'a' to 'b' is present if and only if $a \sqsubseteq b$ and there is no element $x \in S$ such that $a \sqsubseteq x$ and $x \sqsubseteq b$.

Example: Let $S_1 = \{0, 1, 2\}$ and assume $X = P(S_1)$.Also take $\subseteq$ as a relation on $X(= P(S_1))$.This relation is anti-symmetric, transitive and reflexive. So it is a partial order relation on X.

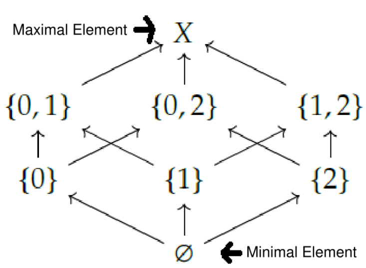

Figure 1: Hasse Diagram for the power the set of {0,1,2}

Definition 6 (Maximal and Minimal Element) *Let $(S, \leq)$ is a poset (i.e.'$\leq$' is a partial order relation on S).*

- *$x \in S$ is said to be a **maximal element** of S if $x \nleq a$ for all $a \in S$ and $a \neq x$.*
- *$y \in S$ is said to be a **minimal element** of S if $b \nleq y$ for all $b \in S$ and $b \neq y$.*

Note that minimal element and minimum element are different. Similarly maximal and maximum are also different. Moreover both minimal and maximal may not be unique. In Figure 1, $\emptyset$ and X are minimal and maximal elements respectively. Let us take one more example. Suppose $S = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$ and take the relation '$\subseteq$' on S. Clearly this is a Partial order relation.

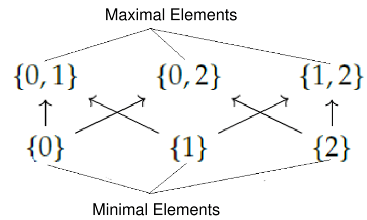

Figure 2: Hasse Diagram for $S = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$

Figure 2, shows that maximal and minimal may not be unique. $\{0, 1\}, \{0, 2\}, \{1, 2\}$ are the maximal elements and $\{0\}, \{1\}, \{2\}$ are the minimal elements.

Definition 7 (Total Order) *If a relation is partial order as well as connex then that relation is said to be Total Order. So in this order we can compare any two elements of the set. Such as the relation '$\leq$' on the set $\mathbb{R}$ is a total order.*

Definition 8 (Well Order) *A well order on a set S is a total order on S with the property that every non-empty subset of S has a least element in this ordering. So $\mathbb{N}$ has well order with respect to the relation $\leq$ because '0' is the least element of $\mathbb{N}$.*

Definition 9 (Lexicographic Order/ Dictionary Order) *If A has a total ordering then the lexicographic ordering is defined on A^n, for some $n > 0$ as follows:*
for any $\bar{X} = (x_1, x_2, ..., x_n), \bar{Y} = (y_1, y_2, ..., y_n) \in A^n, \bar{X} \leq \bar{Y}$ if and only if $x_i \leq y_i$ for the first index i such that $x_i \neq y_i$.

• A set of n-length binary strings can be ordered by using lexicographic ordering. Similarly, $\{0, 1\}^n$ can also be ordered.

Problem 3 *Prove that Lexicographic order is a total order.*

2 Function

Definition 10 (Function) *For two nonempty sets A and B, a relation f from A into B(i.e. $f \subseteq A \times B$) is called a* ***Function*** *from A to B if*

- *(i) domain of $f = A$.*
- *(ii) f is well defined (or, single valued) in the sense that for all $(a, b), (a', b') \in f, a = a'$ implies that $b = b'$, i.e.*

$$a = a' \implies f(a) = f(a').$$

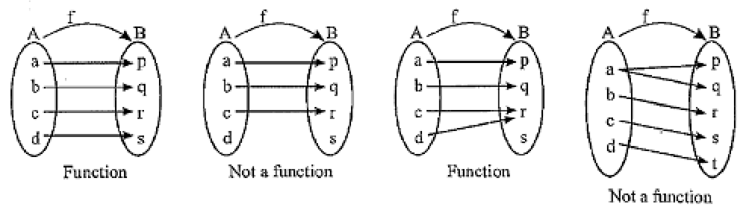

Figure 3:

Example: Let f be the subset of $\mathbb{Z} \times \mathbb{Z}$ defined by $f = \{(n, 4n-5) : n \in \mathbb{Z}\}$. Then domain of f, $D(f) = \{n : n \in \mathbb{Z}\} = \mathbb{Z}$. To show that f is well-defined, let $n = m$ for some $n, m \in \mathbb{Z}$. Then $4n - 5 = 4m - 5$, where $f(n) = f(m)$ and hence f is well-defined. Consequently f is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

Definition 11 *Let us consider a function $f : A \to B$. Then,*

1. *f is called **injective** (or, **one-one**) when for all $a_1, a_2 \in A$ if*

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

(i.e. distint elements of the domian are mapped to the distinct images).

2. *f is called **surjective** (or, **onto**) if $Im(f) = B$,(i.e. for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$).*
3. *f is called **bijective** if f is both surjective and injective.*

Observation: If $f : A \to B$ is one-one, then $|A| \leq |B|$ (where, $|A|$ denotes the cardinality or size of the set A).

Theorem 2.1 *Prove that $|\mathbb{Z}| = |\mathbb{N}|$.*

Proof.

- First we will prove that $|\mathbb{N}| \leq |\mathbb{Z}|$. Let us consider a function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = x$. So f is injective hence from the above observation $|\mathbb{N}| \leq |\mathbb{Z}|$.
- Next we will prove that $|\mathbb{Z}| \leq |\mathbb{N}|$. Suppose $g : \mathbb{Z} \to \mathbb{N}$ defined by

$$g(x) = \begin{cases} 2|x| & x \geq 0 \\ 2|x| + 1 & x < 0 \end{cases}$$

  So g maps the positive integers into even numbers and negative integers into odd numbers. 0 also going to 0 under the map g. This function is injective which implies that $|\mathbb{Z}| \leq |\mathbb{N}|$.
- from the above two we can conclude that $|\mathbb{Z}| = |\mathbb{N}|$ which completes the proof.

□

Actually in the above proof we are using a very popular theorem which is named as '**Schröder-Bernstein theorem**'. The statement of that theorem is **"If there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$."(i.e. if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.)**

Definition 12 *A **Countable Set** is set with the same cardinality as some subset of the set of natural numbers.*

So a finite set is a countable set. And we proved in the previous theorem that $\mathbb{Z}$ has same cardinality with $\mathbb{N}$ then $\mathbb{Z}$ is countable set.

Problem 4 *Prove that $\mathbb{Q}$ is countable.*

- $\mathbb{R}$ is not countable.(By using '**Cantor Diagonaliztion**')