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Combinatorics

Lecture 10: Combinatorics

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1 How Many ways can you pick k objects from n distinct objects?

1.1 If the ordering among the items matter

We can choose the first object in n ways. If the first object is picked then the second object can be picked in $n-1$ ways and so on.

- Total no of ways to pick:

$$n(n-1)(n-2)....(n-k+1) = \frac{n!}{(n-k)!}$$

1.2 If the ordering among the items don't matter

We can order$k$ distinct items in $k!$ ways. So in the total no of ways, each $k!$ ordering of the objects will correspond to only 1 arrangement.

- Total no of order independent ways to pick:

$$\frac{n!}{(n-k)!k!}$$

- We Denote $\frac{n!}{k!(n-k)!}$ with a special symbol: $\binom{N}{k}$

$$\frac{n!}{k!(n-k)!} = \binom{N}{k}$$

1.3 One more proof with a different technique

We want to choose k elements out of n. For each of the n elements, we can either choose it or not choose it based on the constraint that a total of k elements is chosen out of n.

- Lets take a look at the following expression:
  $(x^0 + x^1)(x^0 + x^1)(x^0 + x^1)$... n such elements

In the resultant of previous expression the coeffiecient of x^k represents the number of ways we can choose k items out of n. Because in the expression $(x^0 + x^1)(x^0 + x^1)(x^0 + x^1)$... from each term we can either choose

- x^0 : equivalent to not choosing that item or
- x^1 : equivalent to choosing that item.

So the coefficient of x^k would be the number of ways we can choose exactly k items out of n.

The expression is called generating function. Before learning more about them, we will take a look at generalized binomial theorem which will make our lives easier when dealing with generating functions.

2 Generalized Binomial Theorem

Generalized binomial theorem allows real exponents other than nonnegative integer. In this the coefficient of x^k in the expansion of $(1+x)^n$ is

$$\frac{n(n-1)...(n-k+1)}{k!}$$

2.1 example:

$$(1+x)^{-1} = 1 + \frac{(-1)}{1!}x^1 + \frac{(-1)(-1-1)}{2!}x^2 + ... = 1 - x + x^2 - x^3 + ... = \sum_{i=0}^{\infty}(-1)^i x^i$$

3 Generating functions

if $a_0, a_1, a_2, ...$ is a sequence and f is a one to one function. Then $\sum_{i=0}^{\infty} f(a_i)x^i$ is a generating function.

3.1 Problem

I have to pay 100 rupees to someone. I have an adequate supply of coins/notes of denomination 1,5,10 and 20 rupees. In how many ways can I pay up?

3.2 Solution

Let a_k be number of ways I can Pay k rupees. We want to find a_{100}. We will try to find this number through genereting functions.

The generating function for this problem would be:

$$Q(x) = (1 + x + x^2 + x^3 + ...)(1 + x^5 + x^{10} + ...)(1 + x^{10} + x^{20} + ...)(1 + x^{20} + x^{40} + ...)$$

- Any term in $Q(x)$ is a product of 4 terms. Exactly one from each set of numbers.
- Here the terms first in pair of parenthesis represent the amount I can pay with 1 rupee. For example, if I choose x^{10} from here then it means I am paying 10 rupess using 1 rupee coin.

- The terms in the second pair of parenthesis represent the amount I pay with 5 rupee coin. If I choose 1 from here, then it means that I am not using any 5 ruppe coin to pay the amount.
- The 3rd and fourth set of numbers represent the amount I am paying with 10 rupee and 20 rupee note respectively. If I choose x^{10} from 3rd set of numbers and x^{80} from 4th set of number, Then it would mean that I am paying one 10 rupee note and four 20 rupee notes.

now,

$$Q(x) = (1 + x + x^2 + x^3 + ...)(1 + x^5 + x^{10} + ...)(1 + x^{10} + x^{20} + ...)(1 + x^{20} + x^{40} + ...)$$

$$Q(x) = (1 - x)^{-1}(1 - x^5)^{-1}(1 - x^{10})^{-1}(1 - x^{20})^{-1}$$

We need to find coefficient of x^{100} in the result and that will be our answer.

4 Another generating function for the previous problem and why it is wrong

Another generating function for this problem could be

$$Q(x) = (1 + x + x^5 + x^{10} + x^{20})^{100}$$

This generating function seems to work because we can choose to pay with Rs. 1, 5, 10 or 20 (or none) each time untill we pay 100 rupees. But the problem with this function is that the ordering is also considered in this generating function. So we may end up overcounting.

For example:

$$5 + 20 + 10 + 5 + 0 + 0 + ...$$

would be different from

$$5 + 5 + 10 + 20 + 0 + 0 + ...$$

In this generating function ordering matters. So we will be counting more if we calculate the coefficient of x^{100} using this generating function.

5 balls and bins problem

5.1 question

How many ways we can put n distinct balls into k distinct bins so that the order of putting the balls into bin matters?

5.2 answer

We will represent the balls as numbers and the bins as * to separate. For example for 5 balls and 3 bins one ordering would be:

1 3 5* * 2 4

it denotes balls 1,3,5 went into box 1, none into box 2 and balls 2,4 went into box 3.

- Notice that the empty space represents that no balls went into box 2.
- here we keep the ordering property conserved. For example : 1 5 3* * 2 4 is different from 1 3 5* * 2 4.
- To represent k bins we only need k-1 '*' s.

With this representation the problem boils down to: how many ways you can arrange n numbers and $k-1$ '*' s. In other words, in how many ways you can arrange n distinct items and $k-1$ identical items?

The answer =

$$\frac{(n+k-1)!}{(k-1)!}$$