Discrete Mathematics 14 Sept 2018

Lecture 13: Fibonacci Number

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1 Fibonacci Number

The n'th fibonacci number denoted by F_n is defined recursively as

$$F_n = F_{n-1} + F_{n-2} \qquad\qquad n \geq 2$$

$F_0 = 0$ and $F_1 = 1$.
The golden ratio is the ratio of of two consecutive fibonacci number F_{n+1} and F_n for $n \to \infty$.

$$\textit{Golden Ratio} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

2 Generating Function for Fibonacci number

$$\begin{aligned}
F(X) &= \sum_{i=0}^{\infty} F_i X^i \\
&= 0 + X + \sum_{i=2}^{\infty} F_i X^i \\
&= 0 + X + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) X^i \\
&= X + \sum_{i=2}^{\infty} F_{i-1} X^i + \sum_{i=2}^{\infty} F_{i-2} X^i \\
&= X + X \sum_{i=2}^{\infty} F_{i-1} X^{i-1} + X^2 \sum_{i=2}^{\infty} F_{i-2} X^{i-2} \\
&= X + X \sum_{i=1}^{\infty} F_i X^i + X^2 \sum_{i=0}^{\infty} F_i X^i \\
&= X + XF(X) + X^2 F(X) \\
\therefore (1 - X - X^2) F(X) &= X \\
F(X) &= \frac{X}{1 - X - X^2} \\
F(X) &= \frac{-X}{X^2 + X - 1} \\
\therefore F(X) &= \frac{\alpha}{X - \phi} + \frac{\beta}{X - \gamma}
\end{aligned}$$

$$\phi\gamma = -1, \qquad \phi + \gamma = -1$$
$$\phi = \frac{-1-\sqrt{5}}{2}, \qquad \gamma = \frac{-1+\sqrt{5}}{2}$$
$$\alpha + \beta = -1, \qquad \alpha\gamma + \beta\phi = 0$$
$$\alpha = \frac{-1-\sqrt{5}}{2\sqrt{5}}, \qquad \beta = \frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$\therefore F(X) = \frac{\alpha}{-\phi}\frac{1}{(1+X\gamma)} + \frac{\beta}{-\gamma}\frac{1}{(1+X\phi)}$$

$$\textit{We know, } \frac{1}{(1+\phi X)} = \sum_{i=0}^{\infty}(-1)^i(\phi X)^i$$

$$\textit{and } \frac{1}{(1+\gamma X)} = \sum_{i=0}^{\infty}(-1)^i(\gamma X)^i$$

$$\therefore F(X) = -\frac{1}{\sqrt{5}}\sum_{i=0}^{\infty}(-1)^i(\gamma X)^i + \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty}(-1)^i(\phi X)^i$$

$$\Rightarrow F(X) = \sum_{i=0}^{\infty}(\frac{-1}{\sqrt{5}} \cdot (-1)^i \cdot \gamma^i + \frac{1}{\sqrt{5}} \cdot (-1)^i \cdot \phi^i)X^i$$

$$\therefore F_n = \frac{-1}{\sqrt{5}} \cdot (-1)^n \cdot \gamma^n + \frac{1}{\sqrt{5}} \cdot (-1)^n \cdot \phi^n$$

$$\Rightarrow F_n = (-1)^n \cdot \frac{\phi^n - \gamma^n}{\sqrt{5}}$$

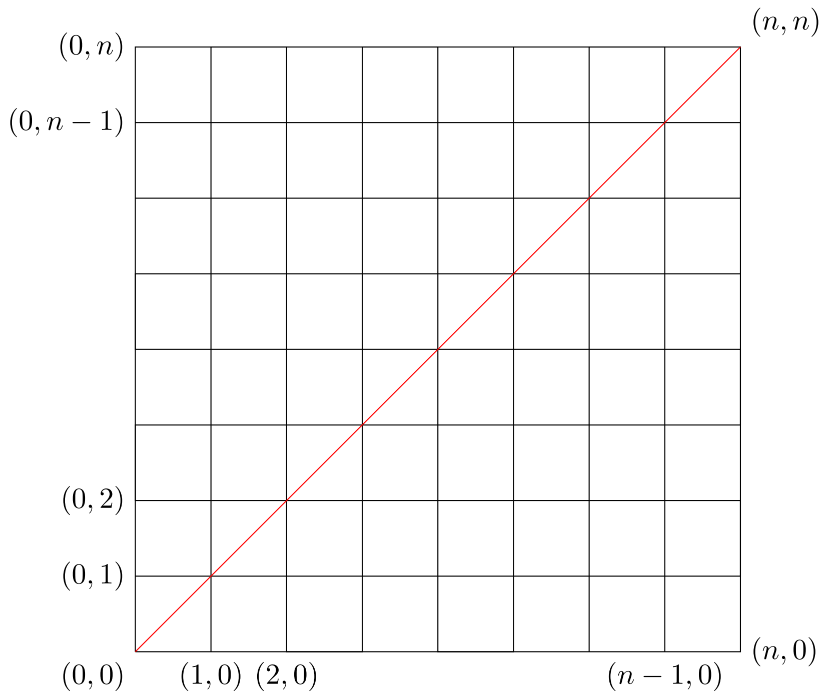
$$\Rightarrow F_n = \frac{\phi^n - \gamma^n}{\phi - \gamma}$$

3 Catalan Number

How many shortest path are there to go from point $(0,0)$ to (n,n) without crossing the Wall ? We can move only $\rightarrow$ and $\uparrow$.
Let e_n= Set of shortest paths from $(0,0)$ to (n,n) such that the diagonal is not crossed.
Let e_n^i= Set of valid shortest paths from $(0,0)$ to (n,n) such that each path 1st time touches the Wall at (i,i).
Clearly, $|e_n^n| = |e_{n-1}|$ and

$$e_n = \bigcup_{i=0}^{n} e_n^i. \tag{1}$$

$$\therefore |e_n| = \sum_{i=0}^{n} |e_n^i| \qquad \textit{as all } e_n^i \textit{ are disjoint} \tag{2}$$



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$$\textit{Now Clearly, } |e_n^i| = |e_{i-1}| \cdot |e_{n-i}| \tag{3}$$

$$\textit{Let } C_i = |e_i| \tag{4}$$

$$\textit{So from (2) and (3), } C_n = \sum_{i=0}^{n} C_{i-1}C_{n-i} \tag{5}$$

$$\therefore C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} \tag{6}$$

4 Generating Function for Catalan Number

$$\begin{aligned}
e(X) &= \sum_{i=0}^{\infty} C_i X^i \\
&= \sum_{n=0}^{\infty} (\sum_{i=0}^{\infty} C_i C_{n-i-1}) X^n \\
(e(X))^2 &= \sum_{n=0}^{\infty} X^n \sum_{i=0}^{n} (C_i C_{n-i}) \\
X(e(x))^2 &= \sum_{n=0}^{\infty} X^{n+1} \sum_{i=0}^{n} (C_i C_{n-i}) \\
Let\ k = n+1 & \\
X(e(X))^2 &= \sum_{k=0}^{\infty} X^k (\sum_{i=0}^{k-1} C_i C_{k-i-1}) \\
&= e(X) - 1 \\
e(X) &= \frac{1 \pm \sqrt{1-4X}}{2X} \\
\lim_{n\to\infty} \frac{1-\sqrt{1-4X}}{2X} = 1\ and & \\
\lim_{n\to\infty} \frac{1+\sqrt{1-4X}}{2X} = \infty & \\
So\ e(X) &= \frac{1-\sqrt{1-4X}}{2X}
\end{aligned}$$

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\begin{aligned}
\sqrt{1-4X} &= \sqrt{1+y} \\
&= (1+y)^{1/2} \\
&= \sum_{n=0}^{\infty} \binom{1/2}{n} y^n \\
&= \sum_{n=0}^{\infty} y^n \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdot(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} \\
&= \sum_{n=0}^{\infty} y^n \frac{(-1)^{n-1}}{2^n}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!} \\
&= \sum_{n=0}^{\infty} y^n \frac{(-1)^{n-1}}{2^n(2n-1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{n!} \\
&= \sum_{n=0}^{\infty} y^n \frac{(-1)^{n-1}}{2^{2n}(2n-1)}\cdot\frac{2^n\cdot n!\cdot 1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{(n!)^2} \\
&= \sum_{n=0}^{\infty} y^n \frac{(-1)^{n-1}}{4^n(2n-1)}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdots 2n}{(n!^2)} \\
&= \sum_{n=0}^{\infty} y^n \frac{(-1)^{n-1}}{4^n(2n-1)}\cdot\binom{2n}{n}
\end{aligned}
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