

Lecture 2: Logic, Intro to Proof Techniques

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1 Logic

1.1 Categories of proofs

1. Theorem

Theorem is a set of assumptions to make some deductions.

$$(\text{Set of Assumptions}) \Rightarrow \text{Deductions}$$

2. Proof

Proofs are categorized in the following two types :

(a) Empirical Proof

Sample of some assumptions from set of all assumptions is used to imply that deduction holds for set of all assumptions.

This type of proof is comparatively easy and can be erroneous.

(b) Mathematical Proof

For all assumptions in the set of assumptions the deduction hold true.

This type of proof is often complicated but is always correct.

Note : Statistical Proof belongs to the category of Mathematical Proof.

3. Examples:

(a) Previously to generate prime number the formula : $n^2 - n + 41$ was used. It was proved using Empirical Proof.

As one can see, this formula does not hold true for $n = 41$

(b) Between 1930 - 1940, 2 biologist came up with perceptron learning algorithm and had provided an empirical proof. Only in 1980 it was mathematically proved.

1.2 Propositional Logic

1. Axiom

Every Statement that is either True or False is proposition.

Proposition are linked using connections.

(a) Examples:

- i. Every city in India has horrible traffic. Kolkata is an Indian City. So Kolkata has horrible traffic.
- ii. Every city in India has good traffic. Kolkata is an Indian City. So Kolkata has good traffic.

2. Truth Table

Truth Table of a statement S is a function f from some domain D to Range R

$$\forall x \in D : f(x) \in R$$

If Truth Table of two different statements are same then those statements are congruent (represented as \equiv) to each other.

3. Connections

The different types of connections are as follows,

- (a) Negation (Represented as \neg)

For proposition p, the truth table for $\neg p$ would look like,

p	$\neg p$
True	False
False	True

- (b) And (Represented as \wedge)

For proposition p and q, the truth table for $p \wedge q$ would look like,

p	q	$p \wedge q$
True	True	True
True	False	False
False	True	False
False	False	False

- (c) Or (Represented as \vee)

For proposition p and q, the truth table for $p \vee q$ would look like,

p	q	$p \vee q$
True	True	True
True	False	True
False	True	True
False	False	False

- (d) Implies (Represented as \Rightarrow)

For proposition p and q, the truth table for $p \Rightarrow q$ would look like,

p	q	$p \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Think of it like,

- i. From a set of True deductions, one will never end up with a False statement.
- ii. A False Statement can lead to anywhere.

Example:

- i. If $2 + 2 = 5$, prove that you are Pope!

Proof. $2 + 2 = 5$

$\Rightarrow 4 = 5$

$\Rightarrow 1 = 2$

$\Rightarrow 1 \text{ person} = 2 \text{ person}$

$\Rightarrow 1 \text{ person (me)} = 2 \text{ person (me and Pope)}$

Therefore, we can see that a false statement can lead anywhere. □

Note : The Truth Table of $\neg p \vee q$ is same as $p \Rightarrow q$, hence

$$\neg p \vee q \equiv p \Rightarrow q$$

- (e) If and Only If (Represented as \Leftrightarrow)

For proposition p and q, the truth table for $p \Leftrightarrow q$ would look like,

p	q	$p \Leftrightarrow q$
True	True	True
True	False	False
False	True	False
False	False	True

Note : The Truth Table of $(\neg p \vee q) \wedge (\neg q \vee p)$ is same as $p \Leftrightarrow q$, hence

$$(\neg p \vee q) \wedge (\neg q \vee p) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv p \Leftrightarrow q$$

For a theorem (statement) $A \Rightarrow B$, to be True,

$A \Rightarrow B$ should be True for all combination of values of variables of A bounded by the bounds defined in the theorem.

From Example 1(a)i, It is a proposition made up of three propositions.

$$\begin{array}{c}
 \overbrace{q(x)=\text{True} \forall x \in \text{Indian Cities}} \\
 \text{Every city in India has horrible traffic.} \\
 \overbrace{Kolkata \in \text{Indian Cities}} \\
 \text{Kolkata is an Indian City.} \\
 (q(x)=\text{True} \forall x \in \text{Indian Cities}) \wedge (Kolkata \in \text{Indian Cities}) \Rightarrow q(\text{Kolkata}) = \text{True} \\
 \text{So Kolkata has Horrible traffic.}
 \end{array}$$

From Example 1(a)ii, It is a proposition made up of three propositions.

$$\overbrace{p(x)=\text{True} \forall x \in \text{Indian Cities}} \\
 \text{Every city in India has good traffic.}$$

$$\begin{array}{c}
 \overbrace{Kolkata \in Indian\ Cities} \\
 \overbrace{Kolkata\ is\ an\ Indian\ City.} \\
 (p(x)=True \ \forall x \in Indian\ Cities) \wedge (Kolkata \in IndianCities) \Rightarrow p(Kolkata) = True \\
 \overbrace{So\ Kolkata\ has\ Good\ traffic.}
 \end{array}$$

1.3 Predicate Logic

1. Definition:

Every Statement or Proposition or a function where range is {True, False}. And the variables must be quantified.

(a) Examples of Predicate Logic are

- i. $(x^2 - 1)$ is divisible by 4 : $\forall x, x \in \mathbb{R}$
- ii. $(x^2 - 1)$ is divisible by 4 : $\exists x, x \in \mathbb{R}$

Note :

(a) $\neg (\forall x P(x)) \equiv (\exists x \neg P(x))$

(b) $\neg (\exists x P(x)) \equiv (\forall x \neg P(x))$

2. Laws:

(a) Associative :

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

(b) Commutative :

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

(c) Distributive :

$$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$$

$$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$$

(d) De Morgan's :

$$\neg (p \vee q) \equiv (\neg p \wedge \neg q)$$

$$\neg (p \wedge q) \equiv (\neg p \vee \neg q)$$

2 Introduction to Proof Techniques

The following are some of the Proof Techniques that can be used to prove $A \Rightarrow B$

- 1. Direct Proof
- 2. Contradiction

(a) Example

i. Instead of proving $A \Rightarrow B$ we can prove $(\neg B \wedge A) \Rightarrow \text{False}$

This is because ,

$$A \Rightarrow B \equiv ((\neg B \wedge A) \Rightarrow \text{False})$$

3. Contrapositive

(a) Example

i. Instead of proving $A \Rightarrow B$ we can prove $(\neg B \Rightarrow \neg A)$

This is because ,

$$A \Rightarrow B \equiv (\neg B \Rightarrow \neg A)$$

4. Induction

5. Counter Example

3 Assignment Problem

1. Prove that number of prime numbers are infinite.

Hint: All the theorems are of the form $A \Rightarrow B$, where A is the set of assumptions. In this case, the set of assumptions A are all the axioms (like definition of prime number, etc.) , which are not explicitly stated. Hence A is said to be hidden.

Proof.

(We shall be using Contradiction Proof)

$A \Rightarrow B$, where A is hidden and A is always True (A is set of axioms, which are always True)

So by Contradiction, we need to prove $\neg B = \text{False}$

Let us assume number of primes is finite.

$\Rightarrow \exists$ a maximum prime number , let it be p_{max}

\Rightarrow Let all the primes be $p_1, p_2, p_3, \dots, p_{max}$

\Rightarrow Consider $N = (p_1 * p_2 * p_3 * \dots * p_{max}) + 1$

\Rightarrow Now $i \in (1, max)$, $\forall p_i$ does not divide N

\Rightarrow There are two possibilities

(a) N must be a prime and $N > p_{max}$

(b) \exists a prime $\notin [p_1, p_2, p_3, \dots, p_{max}]$

Both the possibilities lead to existence of a prime number larger than p_{max}

\Rightarrow Hence, our assumption that number of primes are finite is false.

Hence, $\neg B = \text{False}$

□

4 Homework

1. Write the contrapositive statement of the following :

For Every α and d where $0 < \alpha \leq 1$, $\exists \beta$

$\beta(\alpha, d)$ has the following property :

Let c_1, c_2, \dots, c_n be a convex set in \mathbb{R}^d where $n \geq d + 1$.

If at least $\alpha * \binom{n}{d+1}$ of the collection of the subfamilies of size $d+1$ has a non-empty intersection then \exists a point in at least $\beta * n$ sets.