

Lecture 4: Mathematical Induction

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1 Induction

Definition 1 *Mathematical induction* is a proof method by which an infinite set is split up into various countable sets to prove a fact about the entire set. An inductive proof has 3 main components

1. **Base Step:** This is the step that provides the basis for the proof on which the other proof is built on.
2. **Inductive Hypothesis:** In this step, The statement in question is assumed correct for a particular n .
3. **Inductive Step:** In this step, The inductive hypothesis is made use to prove that the statement is also true for any other X (for example $n+1$, $2n$, $n-1$ etc.), thus giving us a connection $P(n) \implies P(X)$

The Inductive Hypothesis and the Inductive Step must be chosen carefully. The base step depends upon the choice of Inductive hypothesis.

Problem 1 Prove that $\forall n \in \mathbb{N}$, $1 + 2 + \dots + i = \frac{i(i+1)}{2}$

Proof. Here let $S(i) = \sum_{n=1}^i n$

$$S(1) = \sum_{n=1}^1 n = 1$$

By the Hypothesis,

$$S(1) = \frac{1 \cdot 2}{2} = 1$$

$P(1)$ is true \rightarrow **Base Case**

Assume for some $n \in \mathbb{N}$, $P(n)$ is true. \rightarrow **Induction Hypothesis**

$$S(n) = \sum_{i=0}^n i = \frac{n(n+1)}{2}$$

To Prove: $P(n) \implies P(n+1) \rightarrow$ **Inductive Step**

Lemma 1.1 *If $P(1)$ is true and $\forall i \geq 1 \ \& \ i \in \mathbb{N}, P(i) \implies P(i+1)$ (Where $P(i)$ is the hypothesis for i), Then $\forall i \in \mathbb{N} \ \& \ i \geq 2, P(i)$ is true.*

Proof. (Proof by Contradiction)

Let us assume that the statement is false. Let us consider a set S ,

$$S = \{i \in \mathbb{N} : P(i) \text{ is not true}\}$$

Now consider $x = \min\{i : i \in S\} \dots (1)$

$$\implies x - 1 \notin S$$

$$\implies P(x-1) \text{ is true}$$

$$\implies P(x) \text{ is true (} \because P(i) \implies P(i+1) \text{) } \implies x \notin S \text{ (By Definition of } S \text{)}$$

But by (1) , $x \in S$.

\therefore By Contradiction , if $\forall i \in \mathbb{N} , P(i) \implies P(i+1)$, then $P(i)$ is true $\forall i \in \mathbb{N}$ □

$$S(n+1) = \sum_{i=0}^{n+1} i = \left(\sum_{i=0}^n i \right) + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1) \left(\frac{n}{2} + 1 \right)$$

$$S(n+1) = \frac{(n+1)(n+2)}{2}$$

$\therefore P(n+1)$ is true if $P(n)$ is true.

By Lemma 1.1, We know that if $P(1)$ is true and $P(n) \implies P(n+1)$, then $P(n)$ is true $\forall n \in \mathbb{N}$ □

2 Forms of Induction

2.1 Strong Induction

Strong induction makes the inductive step easier to prove by using a stronger hypothesis: one proves the statement $P(m+1)$ under the assumption that $P(n)$ holds for all natural n less than $m+1$; by contrast, the basic form only assumes $P(m)$. The name "strong induction" does not mean that this method can prove more than "weak induction", but merely refers to the stronger hypothesis used in the inductive step; in fact the two methods are equivalent (Source : Wikipedia)

Example:

If $P(1)$ is true and $\forall i > 1 \ \& \ \forall j < i , P(j)$ is true $\implies P(i+1)$ is true, then $\forall i \in \mathbb{N} , P(i)$ is true.

2.2 Weak Induction

Weak Induction is usually the method by which we try to prove a hypothesis by assuming any one case is true. This induction is limited in the sense, that we have information about

only one case. As mentioned earlier, this is similar to "Strong Induction", in terms of method, But has a weaker hypothesis when compared with the strong one.

2.3 Other Common types

There are induction hypotheses, which may be commonly used. For example, $P(n) \implies P(n+2)$, $P(n) \implies P(2n)$ etc. These variants also have their strong versions.

2.4 Recommended Reading

Transfinite Induction

3 More Examples of Induction

Theorem 3.1 *Prove that A.M. is always greater than or equal to G.M.*

$$\forall n \in \mathbb{N}, \forall a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$$

$$\frac{\sum_{i=0}^n a_i}{n} \geq \sqrt[n]{\prod_{i=0}^n a_i}$$

Proof.

Let the statement be $P(n)$.
 For $P(2)$, To Prove, $\frac{a+b}{2} \geq \sqrt{ab}$
 $\forall a, b \in \mathbb{R}$, we know that

$$\begin{aligned} (a-b)^2 \geq 0 &\implies a^2 + b^2 - 2ab \geq 0 \\ \implies a^2 + b^2 + 2ab &\geq 4ab \implies (a+b)^2 \geq 4ab \\ \implies (a+b) &\geq 2\sqrt{ab} \implies \frac{a+b}{2} \geq \sqrt{ab} \quad \dots (1) \end{aligned}$$

$\therefore P(2)$ is true

Now using $P(2)$, we can show $P(4)$ is also true.

For $P(4)$, To Prove $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

$$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2}$$

From (1) we can get that,

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)} \geq \sqrt{(\sqrt{ab})(\sqrt{cd})} = \sqrt[4]{abcd}$$

So,

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

$\therefore P(4)$ is True.

Proposition 3.2 Applying the same logic, we can find that $\forall i \in \mathbb{N}, P(i) \implies P(2i)$

Lemma 3.3 In the above problem, it can also be proved that $P(i) \implies P(i-1)$.

Proof.

$$P(i) \text{ is true} \implies \frac{a_1 + a_2 + a_3 + \dots + a_i}{i} \geq \sqrt[i]{a_1 a_2 a_3 \dots a_i} \quad \dots(2)$$

$$\text{Assume } a_i = \frac{a_1 + a_2 + a_3 + \dots + a_{i-1}}{i-1}$$

Assuming a_i does not affect the proof's generality as we are trying to prove the lemma only for the first $n-1$ elements.

$$\frac{a_1 + a_2 + a_3 + \dots + a_{i-1} + \frac{a_1 + a_2 + a_3 + \dots + a_{i-1}}{i-1}}{i} = \frac{i(a_1 + a_2 + a_3 + \dots + a_{i-1})}{i(i-1)} = A_i$$

$$A_i = \frac{(a_1 + a_2 + a_3 + \dots + a_{i-1})}{(i-1)} = a_i$$

From (2),

$$A_i \geq \sqrt[i]{a_1 a_2 a_3 \dots A_i}$$

$$A_i^i \geq a_1 a_2 a_3 \dots A_i$$

$$A_i^{i-1} \geq a_1 a_2 a_3 \dots a_{i-1}$$

$$A_i \geq \sqrt[i-1]{a_1 a_2 a_3 \dots a_{i-1}}$$

$$\frac{(a_1 + a_2 + a_3 + \dots + a_{i-1})}{(i-1)} \geq \sqrt[i-1]{a_1 a_2 a_3 \dots a_{i-1}}$$

So $P(i-1)$ is true. □

Proposition 3.4 If $P(i) \implies P(2i)$ and $P(i) \implies P(i-1)$ and $P(2)$ is true, Then by Induction, $P(i)$ is true $\forall i \in \mathbb{N}$. Proof is left as an exercise. □

Theorem 3.5 Prove that $\forall n \in \mathbb{N}, \exists x, y, z \in \mathbb{N} : x^2 + y^2 + z^2 = 14^n$

Proof. Before starting any induction problem, it is important to decide on the variable to induct on. The variable must be chosen such that the variable can be used to prove by some inductive hypothesis.

Here, the inductive hypothesis that can be proven is $P(n) \implies P(n+2)$, where $P(n)$ is the theorem statement.

Inductive Hypothesis

Assuming $P(n)$ is true.

$$x^2 + y^2 + z^2 = 14^n \quad \dots(1)$$

Inductive Step $P(n) \implies P(n+2)$

R.H.S. of $P(n+2)$ is

$$14^{n+2} = 14^n \times 14^2$$

From (1), we get that

$$14^{n+2} = 14^2(x^2 + y^2 + z^2) = (14x)^2 + (14y)^2 + (14z)^2$$

$x, y, z \in \mathbb{N} \implies 14x, 14y, 14z \in \mathbb{N}$

$\therefore P(n) \implies P(n+2)$.

From this Inductive Step, we can deduce that if $P(1)$ is true, then $P(n)$ is true $\forall 2n+1 : n \in \mathbb{N}$. But to prove that $P(n)$ is true $\forall n \in \mathbb{N}$, we have to prove that $P(2)$ is also true.

Base Cases

$P(1)$ is true, because $1^2 + 2^2 + 3^2 = 14^1$

$P(2)$ is true, because $4^2 + 6^2 + 12^2 = 14^2$

\therefore By Induction, $P(n)$ is true $\forall n \in \mathbb{N}$

□