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Lecture 5: ON INDUCTION AND RAMSEY NUMBERS

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Until now, when we proved assertions by inducting on a natural number, n(say), we began by assuming a statement $P(n)$ to be true and proceeded on to prove $P(n+1)$ to be true by beginning our deduction on $P(n)$. However, in this lecture we are going to see something a bit different.

1 Problem 1

1.1 Statement of the problem:

There is a tournament with n teams $t_1, t_2, \ldots\ldots, t_n$. For each i, j, t_i play with t_j once and either t_i wins or t_j wins. Prove that there exists an ordering $t_{i_1}, t_{i_2}, \ldots\ldots, t_{i_n}$ such that team t_{i_1} defeats t_{i_2}, team t_{i_2} defeats team t_{i_3} and so on.(***N.B.*:** Team t_{i_1} need not defeat team t_{i_3}.)

1.2 *Solution:*

We will first try to reformulate this problem in the language of graphs. Consider a directed graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots\ldots, v_n\}$ where each vertex v_i represents the team t_i and there exists edge (v_i, v_j) if team t_i defeats team t_j. As in each game there is always a winner so this graph is a tournament. Thus, the above problem now becomes equivalent to proving the following proposition:

Every tournament has a directed Hamiltonian path.

We will now try to solve this by inducting on n, the number of vertices. Let $P(n)$ be the assertion that any tournament with n vertices will have a directed Hamiltonian path.

- **Base Case:** We first consider when $n = 2$. Let the winner among the two teams be t_1 and the other one be t_2. Thus we get a directed Hamiltonian path as follows:

  $$v_1, v_2.$$

  Therefore, $P(2)$ is true.

- **Induction Hypothesis:** Let us assume $P(k)$ that is true where $k \in \mathbb{N}$,i.e., every tournament with k vertices will have a directed Hamiltonian path.

- **Inductive Step:** As usual, we are going to prove that if for any $k \in \mathbb{N}$, $P(k)$ is true then $P(k+1)$ is also true. (However, there is something to be noted here very carefully. While in most other proof based on induction we first began with $P(k)$ and then went on to prove $P(k+1)$ by using different mathematical techniques. In this case though, that will not do, as if we begin with a k-vertex tournament, then one needs to prove $P(k+1)$ for every instance of $(k+1)$-vertex tournaments, which make our deduction highly non-trivial and maybe even impossible. Thus, we shall begin with a $(k+1)$-vertex tournament and then go on to prove that $P(k+1)$ holds using $P(k)$ on a k-subgraph of the said tournament.)

Let G be any tournament of $(k+1)$ vertices. Let v be a vertex of G. Consider, $G' := G \setminus \{v\}$. By induction hypothesis, G' has a directed Hamiltonian path. Let us call that $v_1, v_2, \ldots\ldots, v_k$. We will now use this to prove that $P(k+1)$ is true. We will do so by doing the following case studies.

Case(i): That $(v, v_1) \in E(G')$. Then we are done, as we take the following as the directed Hamiltonian path in G:

$$v, v_1, \ldots\ldots, v_k.$$

Case(ii): That $(v_k, v) \in E(G')$. Then we are done, as we take the following as the directed Hamiltonian path in G:

$$v_1, \ldots\ldots, v_k, v.$$

Case(iii): Before we proceed we will need a lemma, proving which is left as an excercise.

Lemma 1.1 *Consider the directed Hamiltonian path $v_1, v_2, \ldots\ldots, v_k$ in the tournament G'. There exists a $j \in \{1, 2, \ldots\ldots, k\}$ such that $\{(v_j, v), (v, v_{j+1})\} \in E(G)$.*

Thus, by the above lemma we have the following directed Hamiltonian path in G:

$$v_1, \ldots, v_j, v, v_{j+1}, \ldots, v_k.$$

Thus we have proved from the above three cases that whenever $P(k)$ is assumed to be true then $P(k+1)$ is also true.

Thus we have shown that $P(k)$ is true for each $k \in \mathbb{N}$ and $k \geq 2$.

2 Problem 2

2.1 Statement of the problem:

In a meeting there are $2n$ people. Some of them shake hands. However, if A shakes hands with B and B shakes hands with C, then A does not shake hands with C. Prove that the number of handshakes is atmost n^2.

2.2 *Solution:*

We will once again use the language of graphs to prove the above assertion. Let $G(V, E)$ be a graph where the set of vertices $V := \{v_1, v_2, \ldots\ldots, v_{2n}\}$ denotes the set of people in the above mentioned meeting, and $(v_i, v_j) \in E$, the edge set of G, if v_i shakes hands with v_j. By the condition given in the problem, G is a triangle-free graph. Thus our problem is equivalent to the following one on graphs:

> *Any triangle-free graph with* $2n$ *vertices,where* $n \in \mathbb{N}$ *can have atmost* n^2 *edges.*

We will now try to solve this by inducting on n, the number of vertices. Let $P(n)$ be the assertion that any triangle-free graphs with $2n$ vertices can have atmost n^2 vertices.

- **Base Case:***(This is left as an excercise.)*
- **Induction Hypothesis:** Let us assume that $P(k)$ holds true for some $k \in \mathbb{N}$, i.e., any triangle-free graph with $2k$ vertices will have atmost k^2 edges.
- **Inductive Step:** Let G be a triangle-free graph with $2(k+1)$ vertices. If G has no edges then we are done vacuously. Now suppose there is an edge (u, v) among the vertices u and v of G. Consider the graph G' such that $G' := G \setminus \{u, v\}$. Then G' has $2k$ vertices and is triangle-free as well, since it is a subgraph of G. Thus, by induction hypothesis, G' has atmost k^2 vertices. Now, we come back to G. As, G is triangle-free so if any of its vertices has an edge with u, then it cannot have an edge with v. Thus, the edge set of G, $E(G)$ is the disjoint union of $E(G')$, $E(\{u, v\}, \{\overline{u, v}\})$ and $\{(u, v)\}$ and so we have:

  $$|E(G)| = |E(G')| + |E(\{u, v\}, \{\overline{u, v}\}| + 1 \leq k^2 + 2k + 1 = (k+1)^2$$

  Thus we have proved that any triangle-free graph on $2(k+1)$ vertices can have atmost $(k+1)^2$ edges, if we assume that any triangle-free graph on $2k$ vertices can have atmost k^2 edges. In other words, $P(k+1)$ is true if $P(k)$ is assumed to be so.

So, we have proved $P(n)$ to be true for each $n \in \mathbb{N}$.

3 Ramsey Numbers

Definition 3.1 *A clique of an graph is a subset of its vertex set such that the induced subgraph is a complete graph.*

Definition 3.2 *An independent set of a graph is a subset of its vertex set such that no two vertices in the subset are adjacent.*

Definition 3.3 *A natural number* n *is called "*(p, q)*-good" if any graph on* n *vertices either has a clique of size* p *or an independent set of size* q*.*

Observation 3.4 *If* n *is "*(p, q)*-good" then* $(n+1)$ *is also "*(p, q)*-good".*

Proof. The proof of this is left as an excercise. □

Definition 3.5 *The minimum integer that is "(p,q)-good" is denoted by $R(p,q)$. Such numbers are called Ramsey numbers.*

Theorem 3.6 *For any p and $q \in \mathbb{N}$, $R(p+1,q+1) \leq R(p,q+1) + R(p+1,q)$.*

Proof. The proof of this is left as an excercise. □