Discrete Mathematics	August 24, 2018

Lecture 9: Recurrences and Counting

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1 Recurrences

1.1 Master's theorem

The Master's theorem can compute bounds for recurrences of the following form:

$$T(n) = aT(n/b) + f(n) \tag{1}$$

and $d = \log_b a$

- case 1: If $f(n) = \mathcal{O}(n^c)$ and $c < d$ then $T(n) = \mathcal{O}(n^d)$
- case 2: If $f(n) = \Theta(n^d \log^k n)$ and $k \geq 0$ then $T(n) = \Theta(n^d \log^{k+1} n)$
- case 3: If $f(n) = \Omega(n^c)$ and $c > d$ and $af(n/b) \leq kf(n)$ then $T(n) = \Theta(f(n))$

Ex: Hand in a proper solution of all cases of Master's theorem

Example: $T(n) = aT(n/b) + 10n^c$

$T(n) = aT(n/b) + 10n^c$
$\implies a^2T(n/b^2) + 10a(n/b)^c + 10n^c$
$\implies a^3T(n/b^3) + 10a^2(n/b^2)^c + 10a(n/b)^c + 10n^c$

Generalizing the above equation
$\implies a^kT(n/b^k) + 10n^c(1 + (a/b^c) + (a/b^c)^2 + ... + (a/b^c)^{k-1}$

let $k = \log_b n$
$T(n) = a^{\log_b n}T(1) + 10n^c((a/b^c)^k - 1)/((a/b^c) - 1)$
$\implies n^{\log_b a} + 10n^c(n^{\log_b (a/b^c)} - 1)/((a/b^c) - 1)$
$\implies n^{\log_b a} + 10n^c(n^{\log_b a - c} - 1)/((a/b^c) - 1)$
$\implies n^{\log_b a} + (10n^{\log_b a} - 10n^c)/((a/b^c) - 1)$

Since $c < d$
$T(n) = \mathcal{O}(n^{\log_b a})$ □

2 Counting

We are going to be covering the following ways of counting in class:

- Straightforward Counting
  eg. The number of ways of selecting k objects from n objects is $\binom{n}{k}$
- Recurrences
- Generating functions

2.1 Balls and bins problems

Problem statement: There are n balls and k bins and I want to put n balls into k bins. In how many ways can one achieve this?

	Balls are indistinguishable	Balls are distinct	
		Ordering matters	Ordering doesn't matter
Bins are distinct			
Bins are not distinct			

Ex: The table has to be filled in as a homework

3 Discussion of quiz questions

3.1 Question 1

Problem statement: Write the negation of the following statement: "For all C,D,E,F $\geq$ 0 there exists an N $\in \mathbb{N}$ such that for all n > N we have, $C2^n > Dn^8$ and $E(\log n)^4 < F(n^{1/100})$."

The first step is to identify the quantifiers, variables and predicates in the statement and then rewrite the statement in terms of the quantifiers, variables as well as predicates.
"For all C,D,E,F $\geq$ 0" is equivalent to $\forall x$
"there exists an N $\in \mathbb{N}$" is equivalent to $\exists y$
"for all n > N" is equivalent to $\forall z$
"$C2^n > Dn^8$ and $E(\log n)^4 < F(n^{1/100})$" is equivalent to P(x,y,z)
So the statement can now be written as $\forall x \exists y \forall z P(x, y, z)$

Note: You cannot flip quantifiers: $\forall x \exists y P(x, y) \neq \exists y \forall x P(x, y)$

The negation of $\forall x \exists y \forall z P(x, y, z)$ is $\exists x \forall y \exists z \neg P(x, y, z)$

So, the negation of the statement in words is as follows: "There exists C,D,E,F $\geq$ 0 for all N $\in \mathbb{N}, \exists n > N$ such that $C2^n \leq Dn^8$ and $E(\log n)^4 \geq F(n^{1/100})$"

3.2 Question 2

Problem statement: Prove that a graph is bipartite if and only if the graph has no odd cycle.

You have to take into consideration that there are two parts to the question:
a) Prove that if a graph is bipartite then the graph has no odd cycles
b) Prove that if a graph has no odd cycles then the graph is bipartite

3.3 Question 3

Problem statement: If $T(n) = 3T(\lceil n/3 \rceil) + 1$ and $T(1) = 1$ then what is $T(n)$?

Assume $T(n) = \mathcal{O}(n)$ and we will prove by induction that $T(n) \leq cn + d$ for some fixed c,d to be decided later.

Base case: $T(1) = 1 \leq c.1 + d$

$T(n) = 3T(\lceil n/3 \rceil) + 1 \neq 9T(\lceil n/9 \rceil) + 1 + 3$

Instead this should be written as follows:

$T(n) = 3T(\lceil n/3 \rceil) + 1$

$\implies 9T(\lceil (\lceil n/3 \rceil)/3 \rceil) + 1 + 3$

$\implies 9T(\lceil n/9 \rceil) + 1 + 3$

since $\lceil (\lceil n/3 \rceil)/3 \rceil = \lceil n/9 \rceil$

Also, it is better to give a lower bound as well as an upper bound for the recurrence you are solving