Assignment 2
Discrete Mathematics - MTech CS 2019

All the problems marked with (*) are a bit hard and may need ideas not necessarily cover in the class so far. But you are encouraged to try the problems before the sollutions are discussed in class.

1. Write the contraposetive of the following statement:

   Given a finite family of convex sets $C_1, C_2, ..., C_n$ in $\mathbb{R}^d$ (where $n \geq d+1$) such that if the intersection of every $d+1$ of these sets is non-empty, then the whole collection has a non-empty intersection.

2. Prove that for any posetive reals a, b

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

3. Prove that for any posetive reals a, b, c, d

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4}.$$

4. Prove that $\sqrt{2}$ is not rational.

5. Prove that $\sqrt{2} + \sqrt{3}$ is not rational.

6. Is the statement $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$ a tautology or a contradiction or none.

   Definition: A proposition that is always TRUE is called a tautology. A proposition that is always FALSE is called a contradiction.

7. Prove that there are infintely many primes of the form $3(mod 4)$.

8. (*) Prove that there are infinitely many primes of the form $1(mod 4)$.

9. Write the opposite of the following statement:

   There is an university in USA where every department that has at least 20 faculty has at least one noble laureate.

10. What is the contrapositive of the statement

    For all $C, D, E, F \geq 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have , $C2^n > Dn^8$ and $E(\log n)^4 < F(n^{1/100})$.

11. For natural number p and q, the Ramsey number $R(p, q)$ is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who don't know each other. Prove that Note that $R(p, 1) = R(1, q) = 1$. Prove that:

    (a) $R(p+1, q+1) \leq R(p, q+1) + R(p+1, q)$

    (b) $R(p, q) \leq C_{p-1}^{p+q-2}$

12. Write the negation of the following statement:

$$\forall x \geq 0 \ \exists y \in \mathbb{N} \quad (y \geq x) \wedge (y \text{ is a prime})$$

13. (a) We know that $\sqrt{3}$ is not rational. Using this prove that $\sqrt{3} + \sqrt{24}$ is not rational.

    (b) If m is a positive integers such that $\sqrt{m}$ is not rational then prove that for any positive integer n the number $\sqrt{m} + \sqrt{n}$ is not rational.

14. A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.

    (a) For any given n, give an example of a tournament which has no directed cycle.

    (b) Prove that a tournament has a directed 3-cycle if and only if it has a directed cycle.

15. The following are three wrong statments and their proofs. Find out where the mistake is in each of the three case.

    (a) Statement: All cows are the same colour.

    Proof. We show by induction that in any collection of n cows all n of them are the same colour.

    Base Case: If $n = 1$ the therom holds.

    Inductive Step: For the inductive step assume that in any collection of n cows all n of them are the same colour. Consider a set $n + 1$ of cows numbered 1 to $n + 1$. By the induction hypothesis cows 1 to n are the same colour and similary cows 2 to $n + 1$ are the same colopur. But the middle cows 2 to n cant change colour according to who they are grouped with so cows 1 to $n + 1$ must all be the same colour. ■

    (b) Statement: For a list of length n, mergesort takes $O(n)$ time.

    Proof. Let $P(n)$ denote the number of steps taken by merge sort to sort n items.

    Base Case: merge sort on the empty list just returns the empty list. So $P(0) = 0$.

    Strong induction: Assume $P(1), ..., P(n-1)$ and try to prove $P(n)$. We know that at each step in a recursive mergesort, two approximately "half-lists" are mergesorted and then "zipped up". The mergesorting of each half list takes, by induction, $O(n/2)$ time. The zipping up takes $O(n)$ time. So the algorithm has a recurrence relation of $M(n) = 2M(n/2) + O(n)$ which is $2O(n/2) + O(n)$ which is $O(n)$. ■

(c) Statement: Every graph with more than 3 vertices and minimum degree 2 contains a 3-cycle.

Proof. We proceed by induction on $|V(G)|$. As a base case, observe that the theorem is true when $|V(G)| = 3$, since any simple graph on three vertices with all vertices of degree ≥ 2 must be a cycle of length 3.

To prove the inductive step, let G be a graph on $n-1$ vertices for which the theorem holds, and construct a new graph G_0 on n vertices by adding one new vertex to G and ≥ 2 edges incident with this new vertex. Since G contained a cycle of length 3, the graph G_0 also contains a cycle of length 3. This completes the proof. ■