Assignment 3
Discrete Mathematics - MTech CS 2019

1. Let $x > -1$ be a real number. Prove that $(1+x)^n \geq 1 + nx$ for all natural numbers n.

2. Prove that $\sum_{i=1}^{n} i \times i! = (n+1)! - 1$.

3. Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n. Prove that $a_n = 2^n + 3^n$ for all natural number n.

4. In a party there are $2n$ participants, where n is a natural number. Some participants shake hands with other participants. It is known that there do not exist three participants who have shaken hands with each other. Prove that the total number of handshakes in not more than n^2.

5. Let $n > 1$ be an integer. In a football league there are n teams. Every two teams have played against each other exactly once, and in match no draw is allowed. Prove that it is possible to number the teams in such a way that team i beats $(i+1)$ for $i = 1, 2, \ldots, n-1$.

6. Prove that $2002^{n+2} + 2003^{2n+1}$ is divisible by 4005.

7. From a pack of 52 playing cards one extract the 26 red cards and pairs them up randomly with the black cards. The back sides of each pair of cards are then glued together, resulting in 13 cards with both sides being "the front". Prove that it is always possible to flip the cards so that the 13 sides facing upwards are $A, 2, 3, \ldots, 10, J, Q, K$.

8. The Fibonacci sequence is defined as $x_o = 0$, $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ for all non-negative integers n. Prove that

   (a) $x_m = x_{r+1}x_{m-r} + x_r x_{m-r-1}$ for all integers $m \geq 1$ and $0 \leq r \leq m-1$;

   (b) x_d divides x_{kd} for all positive integers d and k.

9. A graph $G = (V, E)$ is k colorable if there exist a function from $c : V \to \{1, 2, \ldots, k\}$ such that if $(i, j) \in E$ then $c(i) \neq c(j)$. It is like can we color the vertices with k colors such that no two adjacent vertices has the same color.

   (a) Prove that if a graph has maximum degree less than or equal to k then it is $(k+1)$-colorable.

   (b) Prove that if all vertices in G has degree ≤ 3 and at least one vertex has degree < 3 then the graph is 3-colorable.

10. Prove that a graph is bipartite if and only if the graph has no odd cycle.

11. If G is a graph such that all vertices has degree more that 2 then G has a cycle.

12. If G is a graph such that all the vertices has even degree then prove that G can be written as a union of edge-disjoint cycles.

13. A simple undirected graph where any two vertices are connected is called a **connected graph**. A simple undirected graph which has no cycle is called **acyclic**. An acyclic connected graph is called a tree.

    (a) Prove that every tree has atleast 2 vertex of degree 1.

    (b) Prove that a tree on n vertices has exactly $(n-1)$ edges.

    (c) Prove that is a graph G is connected and has exactly $(n-1)$ edges then G is a tree.

    (d) A tree where every vertex has degree 3 or 1 (except the root that has degree 2 and degree zero is the tree is of size 1) is called a binary tree. Prove that a binary tree on n vertices has at least $\lfloor n/2 \rfloor + 1$ number of leaves. (Hint: induction pivoting at the root).