Lecture 17: Introduction to Graph Theory

Instructor: Sourav Chakraborty *Scribe: Subrat Prasad Panda*

1 Preliminaries

Definition 1.1 (Graphs) *A graph is a tuple* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *where* $\mathbf{V}$ *is a (finite) set of vertices and* $\mathbf{E}$ *is a finite collection of edges.* The set E contains elements from the union of the one and two element subsets of V . That is, each edge is either a one or two element subset of V .For instance, Figure 1.1 is a graph where $V=\{v_1,v_2,v_3,v_4\}$ and E=$\{e_1,e_2,e_3,e_4,e_5,,e_6\}$.[1]

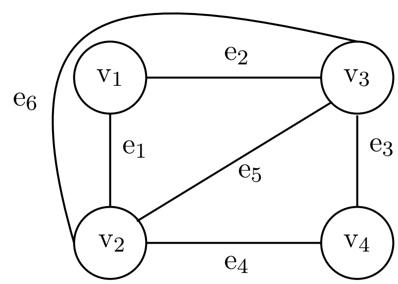


Figure 1.1: Example of a Graph.

Definition 1.2 (Simple Graphs) *A simple graph is a finite undirected graph without loops and multiple edges.[2]*

Definition 1.3 (Degree) *Let* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *be a graph and let* $v \in \mathbf{V}$ *. The degree of v, written deg(v) is the number of non-self-loop edges adjacent to v plus two times the number of self-loops defined at v. More formally:*

$$deg(v) = |\{ e \in E : \exists u \in V \ (e = \{u, v\}) | + 2 |\{e \in E : e = \{v\}\}|$$

Here if S is a set, then $|S|$ is the cardinality of that set.[1]

Definition 1.4 (Path) *A* ***path*** $\mathbf{P}$ *in a graph is a sequence of vertices* $v_1,v_2 ,\ldots, v_k$ *such that* v_iv_{i+1} *is an edge for each i=1 ,..., k-1. The length of a path P is the number of edges in* $\mathbf{P}$*.[2]*

Definition 1.5 (Connected Graphs) *A graph G is* ***connected*** *if any two vertices of the graph are joint by a path. If a graph G is* ***disconnected*** *(i.e., not connected), then every maximal (with respect to inclusion) connected sub-graph of G is called a connected component of G.[2]*

Definition 1.6 (Dense Graphs) *A graph $G = (V, E)$ is said to be dense if for every $v \in V$, $degree(v) > \frac{n}{2}$, where $n = |V|$* i.e number of edges is close to the maximal number of edges or $|E| = \Theta(|V|^2)$.[4]

Definition 1.7 (Sparse Graphs) *A graph $G = (V, E)$ is said to be spare if for every $v \in V$, $degree(v) < \frac{n}{2}$, where $n = |V|$* i.e number of edges is close to the number of vertices or $|E| = \Theta(|V|)$.

2 Representation of Graphs

2.1 Sets and Relation

Any graphs $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ can be represented in forms of Sets and Relation. For instance, the graph in Figure 1.1 can be represented as following:

$$V = \{v_1, v_2, v_3, v_4\}; E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_4, v_2\}, \{v_2, v_3\}, \{v_2, v_3\}\}$$

2.2 Adjacency Matrix

Definition 2.1 (Adjacency Matrix) *Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph and assume that $V = \{v_1, ..., v_n\}$. The adjacency matrix of G is an $n \times n$ matrix A defined as:[1]*

$$A_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & otherwise \end{cases} \tag{2.1}$$

$$A_{ij} = \left(\begin{array}{cc} & \begin{matrix} v_1 & v_2 & \dots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \end{array} \right)_{n \times n}$$

Proposition 2.2 *The adjacency matrix of a simple graph is symmetric with diagonal entries as 0.*

Remark 2.3 *The adjacency matrix representation depends on explicit definition of entries, hence it's not unique. For instance,*

- Using diagonal entries for different purposes(as diagonal entries are futile).
- Representation of zeroed entries with different integers or values.
- In case of simple undirected graph, the adjacency matrix is upper/lower triangular matrix. Hence, definition can be modified to optimize memory usages.

2.3 Adjacency List

Definition 2.4 (Adjacency List) *Let* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *be a graph and assume that* $V = \{v_1, ..., v_n\}$*. The adjacency list of graph* G *is constructed by assigning a unique label from 0 to* $n - 1$ *to each vertex and building an array* A *of length* n *where each entry* $\mathbf{A}[\mathbf{i}]$ *contains a pointer to a linked list of all the out-neighbours of vertex* i*. Figure 2.1 shows the representation of adjacency list of a graph.*

In an undirected graph with edge u, v the edge will appear in the adjacency list for both u and v.

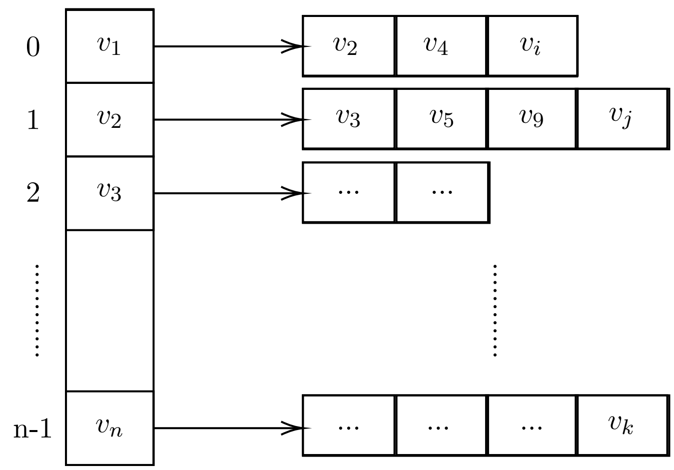


Figure 2.1: Representation of Graph with Adjacency List.

Fact 2.5 *Searching for existence of an edge between any two vertices of a graph takes* $\mathcal{O}(1)$ *in **adjacency matrix** representation, whereas it takes* $\mathcal{O}(\log n)$ *(by the application of binary search) in case of **adjacency list** representation.*

3 Euler's Handshaking Lemma

Theorem 3.1 *In any graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *where* V *and* E *represents vertices and edges then*

$$\sum_{v \in V} \mathbf{deg(v)} = \mathbf{2} \times | \mathbf{E} | \tag{3.1}$$

Proof. It can be proved using Adjacency Matrix. Let a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ represented by Adjacency Matrix A_{ij} where V and E are the vertices and edges respectively. Then $deg(v_i) = \sum_{k=0}^{n-1} A[i][k]$. So, $\sum_{v \in V} deg(v_i) = \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} A[i][k] = 2 \times | E |$ as each edge is represented exactly twice by 1 in the matrix $A[i][j]$, hence, summation of all the elements in matrix $A[i][j]$ results in twice the number of vertices of graph. □

4 Properties of Adjacency Matrix

4.1 Square of Adjacency Matrix

Theorem 4.1 *Let $A = (a_{ij}) = A(G)$ for some simple undirected graph G and define $S = (s_{ij}) = A^2$. Then for every i and j, s_{ij} represents the number of two-walks (walks with two edges) from vertex v_i to v_j in G.[5]*

$$A^2_{ij} = |\ \{\ k \mid A_{ik} = 1 = A_{kj}\ \}\ | \tag{4.1}$$

Proof. Consider the entry s_{ij} in S. By definition, $s_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$ and so one is contributed to the sum only when a_{ik} and a_{kj} are 1. That is, when the edges v_iv_k and v_kv_j are in G, which corresponds to the two-walk, refer Figure 4.2, from v_i to v_j through v_k. □

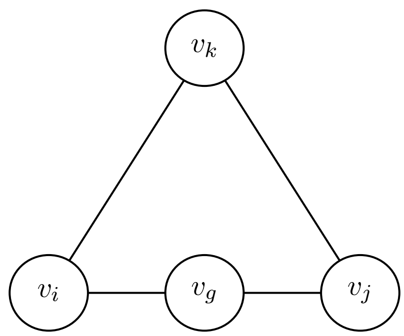


Figure 4.1: Two walks from v_i to v_j through v_k.

Corollary 4.2 *The diagonal elements s_{ii} of $S = A^2$ (where A is the adjacency matrix) is equals to degree of (v_i) i.e*

$$s_{ii} = deg(v_i)$$

4.2 Cube of Adjacency Matrix

Theorem 4.3 *Let A be the adjacency matrix of a simple undirected graph G and define $C_{ij} = c_{ij} = A^3$. Then for every i and j, c_{ij} represents the number of different edge sequences of 3 edges between vertices v_i and v_j. [6]*

$$A^3_{ij} = |\ \{\ (k, l) \mid A_{ik} = 1 = A_{kl} = A_{lj}\ \}\ | \tag{4.2}$$

Proof. Consider the entry c_{ij} in C. By definition, $c_{ij} = \sum_{k=1}^{n} s_{ik}a_{kj}$ which is equals to $\sum_{k=1}^{n}($ number of all different edges sequences of three edges from ith vertex to jth vertex via kth vertex). Refer Figure 4.2, summation produces the number of possible different edge sequences of three edges between ith and jth vertices. □

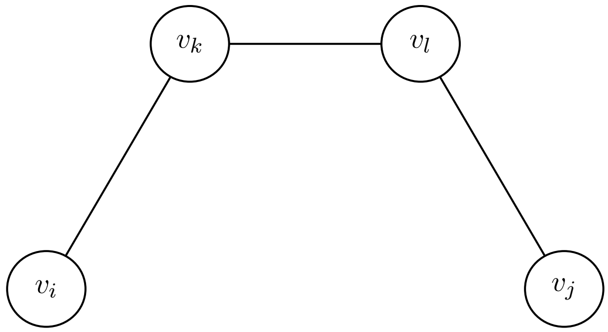


Figure 4.2: Three walks from vertex v_i to v_j through v_k and v_l.

Corollary 4.4 *The diagonals entries $c_{ii} \in A^3$ denotes the number of triangular cycles passing through each vertex i in a simple graph.*

Corollary 4.5 *Trace of A^3 is equals to 3 times the total number of triangular cycles in a simple graph.*

4.3 Eigenvalue of Adjacency Matrix

Definition 4.6 (d-Regular Graph) *A regular graph is a graph where every vertex has same degrees. A regular graph with vertices of degree **d** is called a **d-Regular Graph**.* Figure 4.3 shows examples of 2-Regular Graphs. [7]

(a) 2-Regular Graph with 3 vertices

(b) 2-Regular Graph with 4 vertices

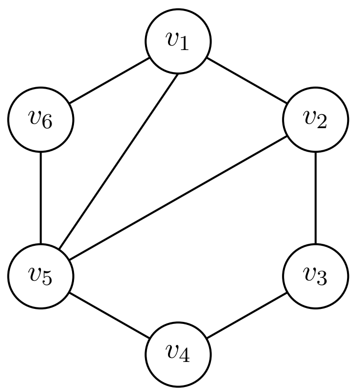
Figure 4.3: Example of d-Regular Graphs

Definition 4.7 (Cut Set) *Cut is a minimal set of edges, removal of which render the graph disconnected. The cut partitions a graph into two or more components making the graph disconnected. Set of edges fulfilling above properties forms cut set.*
Size of cut is defined by the fraction of edges in cut set with respect to cardinality $|E|$ of a graph G(V,E). For instance, $0.1 \times |E|$ is the cardinality of cut set which signifies that the size of cut set is 0.1 times cardinality of Edge.

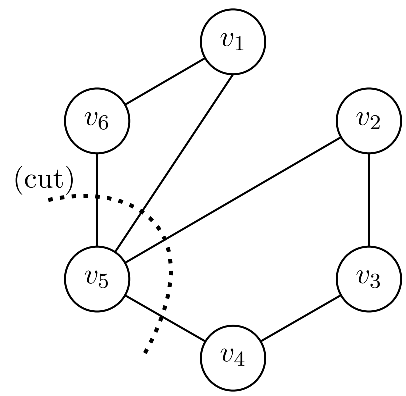
Definition 4.8 (k-Vertex Connected) *A graph G is k-vertex connected if k is the smallest subset of vertices such that the deletion of same renders graph disconnected.* Refer Figure 4.4 for examples.

$$\forall S \subseteq V, |S| \leq k-1 \text{ such that } G(V \setminus S, E) \text{ is connected.}$$

Definition 4.9 (k-Edge Connected) *Let $G = (V, E)$ be an arbitrary graph. If subgraph $G' = (V, E \setminus X)$ is connected for all $X \subseteq E$ where $|X| \leq k - 1$, then G is k-edge-connected. The edge connectivity of G is the maximum value k such that G is k-edge-connected. The smallest set X whose removal disconnects G is a minimum cut in G.* Refer Figure 4.4 for examples.



(a) 4-Edge-Connected and 2-Vertex-Connected Graph



(b) 2-Edge-Connected and 1-Vertex-Connected Graph

Figure 4.4: Example of k-Edge-Connected and k-Vertex-Connected Graph

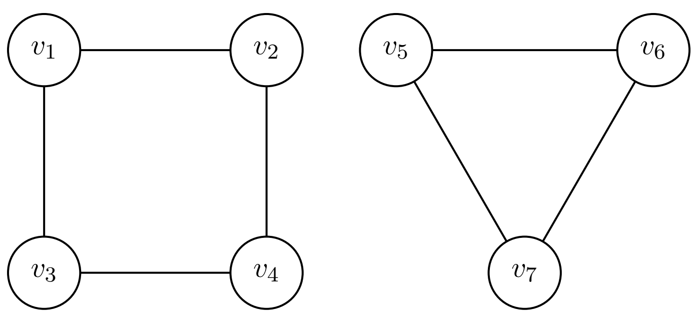


Figure 4.5: A disconnected 2-regular graph G having two components.

Solved Example 1: Find the eigenvalue and eigenvector of graph given in Figure 4.5

Solution: The adjacency matrix A of graph G given in Figure 4.5 is following:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{7\times 7}$$

Hence, the eigenvector and the eigenvalues of adjacency matrix A can be calculated as follows:

$$A \times \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}_{7\times1} = 2 \times \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}_{7\times1} \quad and\ A \times \begin{bmatrix}1\\1\\1\\1\\0\\0\\0\end{bmatrix}_{7\times1} = 2 \times \begin{bmatrix}1\\1\\1\\1\\0\\0\\0\end{bmatrix}_{7\times1}$$

In general for d-Regular-Graph $G(V, E)$, $A \times eigenvector = degree(V) \times eigenvecotor$.

Fact 4.10 *An adjacency matrix of any disconnected graph having two components resembles following matrix, where X and Y are respective adjacency matrix of disconnected components in graph.*

$$\left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right]$$

Claim 4.11 *If a d-regular graph G is represented in adjacency matrix as A_G, then A_G has eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$.*
1. The largest eigenvalue of A_G i.e. $\lambda_1 = degree(V)$.
2. The graph G will be connected iff $\lambda_2 < \lambda_1$. ***Hence, eigenvalue of adjacency matrix of a graph relates to connectivity of graph.***

Corollary 4.12 *Size of min cut in a graph $G \backsim (\lambda_1 - \lambda_2)$.*

5 Cycle Space and Cut Space of Graph

Definition 5.1 (Cycle) *A cycle in a graph $G = (V, E)$ is a list of vertices v_1, v_2,..., v_n, v_1 such that v_i is adjacent to v_{i+1} for all $i \leq n-1$, v_n is adjacent to v_1, and no vertices are repeated.[8]*

Cycle Representation: A cycle C_k of a graph $G = (V, E)$ can be represented as row vector or column vector (array of 0s and 1s) where $C_k \in \{0, 1\}^{|E|}$ and,

$$C_k[i] = \begin{cases} 1 & e_i \in C_k \\ 0 & Otherwise \end{cases} \tag{5.1}$$

For instance, cycles C_1 and C_2 in the graph G shown in Figure 5.6 can be represented as:

$$C_1 = [1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ ]_{1\times10} \tag{5.2}$$

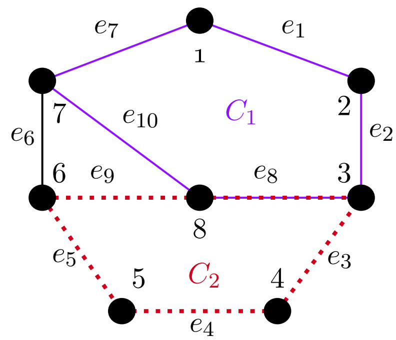


Figure 5.6: Example of cycles in a graph

$$C_2 = [0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ ]_{1\times 10} \tag{5.3}$$

Set of all cycles C_k (row/column vector each) in a graph G forms vector space under ring sum operation known as *Cycle Space*.

Proposition 5.2 (Ring Sum) *Given two graphs $G1 = (V1, E1)$ and $G2 = (V2, E2)$ we define the ring sum $G1 \bigoplus G2 = (V1 \cup V2, (E1 \cup E2) - (E1 \bigcap E2))$ with isolated points dropped.[9]*

From previous example of cycle C_1 and C_2; $C_1 \bigoplus C_2 = [1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1]_{1\times 10}$.

Proposition 5.3 (Cut Space) *Similar to Cycle Space C, Cut Space C^* is defined as set of all cuts(Definition 4.7) under the operation of ring sum.*

Definition 5.4 (Spanning Tree) *A spanning tree T of an undirected graph G is a sub-graph that is a tree which includes all of the vertices of G, with minimum possible number of edges. In general, a graph may have several spanning trees.[10]*

Definition 5.5 (Fundamental Cycle) *Let $G = (V, E)$ be a connected graph and let T be a spanning tree of G. Let an edge $e \in G(E) \setminus T(E)$ between vertices v and w. Now in T there is a unique path between v and w, and since $e \notin T$, that path does not use e. Therefore, that path together with e forms a cycle in G. The cycle formed this way is called a fundamental cycle.*

Definition 5.6 (Fundamental Cut) *Let T be a spanning tree of a connected graph G. Each edge $\boldsymbol{e} \in T$ defines a unique partition of vertices of G. For each $\boldsymbol{e}$ there exist an cut set(c^*) in G rendering the similar partition. The cut sets(c^*) which contains an edge $\boldsymbol{e} \in T$ and few chords of T are known as Fundamental Cut Sets.*

Homework

- To find the basis of Cycle Space and Cut Space.
  Hint: Prove fundamental cycle/cut are independent and spans complete space. Use, number of Fundamental Cycles & Fundamental Cuts are $(|E| - (|V| - 1))$ and $(|V| - 1)$ respectively.
- To proof $C = (C^*)^{\perp}$, where C and C^* are Cycle Space and Cut Space respectively.

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