Midterm
MTech CS Discrete Mathematics, 2019

11th September, 2019

Time: 3 hours, Maximum Marks: 100

Answer as many questions as possible but the maximum possible marks one can obtain is 100. Your answers should be well-written and you should explain your arguments properly.

1. (10 marks) From the definition of the asymptotc notations prove that if $T(n) = aT(n/b) + O(n^c)$, where $c < \log_b a$ then
$$T(n) = \Theta(n^{\log_b a}).$$

2. Prove or disprove the following set of asymptotic relations: (2 marks each)
   (a) $(2.9)^{\log_2 n} = \Theta(n^{\log_2 3})$
   (b) $\log\log n = \Omega((\log\log\log n)^{\log\log\log n})$
   (c) $n^4 \sim (1 - 1/n)^n n^3$
   (d) $n^{10(\log\log n)^{100}} = \Theta((\log n)!)$
   (e) $\log \binom{2n}{n} = o(n^n)$

3. (10 marks) The *Lucas Sequence* $1, 3, 4, 7, 11, 18, 29, \ldots$ is defined by $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$. Prove that $a_n = O(1.75^n)$.

4. (5 + 5 + 5 Marks) Prove or disprove
   (a) Let $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.
   (b) If for a positive interger n, $2^n - 1$ is prime then n is a prime.
   (c) If a and b are two number that are not rational then a^b is also not rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$.)

5. (5 Marks) Prove that at a party with at least two people, that there are two people who know the same number of people there (not necessarily the same people - just the same number) given that every person at the party knows at least one person. (Note that nobody can be his or her own friend.)

6. (3+ 7 Marks) A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.

   (a) For any given n, give an example of a tournament which has no directed cycle.

   (b) Prove that a tournament has a directed 3-cycle if and only if it has a directed cycle.

7. (10 marks) How many ways can you distribute n identical balls into $2k$ distinct bins such that exactly k of the bins gets odd number of balls and the remaining k bins gets even number of balls.

8. (2+3+5 marks) Consider the functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.

   (a) How many functions are there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.

   (b) How many increasing functions are there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.

   (c) How many non-decreasing functions be there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, k\}$.

9. (5+5 marks) Let G be a simple undirected graph.

   (a) Prove that the number of vertices with odd degree in G is even.

   (b) Prove that if u is a vertex of odd degree in a graph, then there exists a path from u to another vertex v of the graph where v also has odd degree.

10. (10 marks) Let $p^*(k)$ be the number of ways to partition the integer k into distinct integers, and $p_o(k)$ be the number of ways to partition integer k into odd integers. Prove that $p^*(k) = p_0(k)$.

11. (10 marks) Matrix multiplication is associative but not commutative. That means $AB \neq BA$ but $(A(BC)) = ((AB)C)$. How many ways you can multiply n matrices $A_1, A_2, \ldots, A_n$ given in that order? You can multiply only two matrices together in your computation.