Discrete Mathematics 22nd July, 2018

Lecture 0: Introductory Lecture

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1 What is Dicrete Mathematics?

Discrete Mathematics is not a subject with well-defined boundaries. Roughly speaking it is the subject that deals with "discrete" objects. What are discrete objects? Anything that is not "continuous" is discrete. A more clear notion of discrete and continuous will develop as the course progresses.

Unfortunately, there is no one definitation of what should be included in discrete mathematics. One should read up Wikipedia and other websites to develop their own interpretation of discrete mathematics. In this course we will cover a bit of logic, combinatorics and graph theory. Infact all these topics can be studied in more depth as seperate courses as is the case for many other topics that is sometimes included is discrete math courses, like, fite set theory, finite graoup theory, finite probability, discrete geometry and many others.

2 Basic Set Theory

A **set** is a collection of distinct objects. The objects that comprise a set are called **elements of the set**. By the notation $x \in S$, we mean "x is an element of the set S." Similarly, By the notation $x \notin S$, we mean "x is not an element of the set S."

The sets are usually representated by listing the elements, seperated by comma, within two curly brackets. Like the set $\{a, e, i, o, u\}$ is the set of the vowels in the english alphabet. Usually there is no ordering among the elements in the set. And usually no element of the set occurs multiple times.

If in a set an element occurs multiple times it is called a **multiset**. And if the ordering of the elements in the set matters then it is called an ordered set or a tuple (if there are only two elements) and n-tuple if there are n elements.

The number of elements in a a set is called the cardinality of a set and is represented by $|S|$, where S is the set.

If T is a set that can be obtained from a set S by throwing away for element then we say T is a subset of the set S and S is a superset of set T, denoted $T \subseteq S$ or $T \subset S$ (if T has at least one element less than the set S).

Some examples of sets are:

- Set of all students attending this class
- Set of all cricketers in the world
- Set of all natural numbers, $\{0, 1, 2, 3, \ldots, \}$, denoted by $\mathbb{N}$
- Set of all integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \}$, denoted by $\mathbb{Z}$

- Set of all real numbers, denoted by $\mathbb{R}$
- Set of all rational numbers, denoted by $\mathbb{Q}$
- Set of all positive real numbers, denoted by $\mathbb{R}^+$.

While the cardinality of sets can be finite and infinite, there are finer divisions in the size of infinite sets. We will discuss that in the next class.

When we talk about sets, usually we assume that there is a universe set, usually denoted Ω, such that the sets in consideration are all subsets of the universe set. The universe set is sometimes explicitly stated while in other times it is implicit.

2.1 Operations of Sets

There are various operations on sets that are done. We will be presenting some of the basic ones that we will need in our course.

1. **Intersection** Given two sets A and B the intersection of A and B, denoted $A \cap B$, is the set of all elements present in both A and B

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

2. **Union** Given two sets A and B the union of A and B, denoted $A \cup B$, is the set of all elements present in either in A or B or both

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

3. **Complement** Given a set A in the universe set Ω the complement of A, denoted A^c or $\bar{A}$, is the set of all elements in the universe Ω but not is A

$$A \cap B = \{x \mid x \in A \text{ or } x \in B\}$$

4. **Cartesan-product** Given two sets A and B the cartesan product of A and B, denoted $A \times B$, is the set of all tuples whose first element is in A and the second element is in B

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

Cartesan product can also be defines between more than 2 sets. Namely, if $A_1, A_2, \ldots, A_n$ are n sets then the cartesan product of the sets, denoted $A_1 \times A_2 \times \ldots \times A_n$ is the set of n-tuples where the ith element in each tuple comes from the set A_i.

The cartesan product of A and A is also denoted as A^2. Similarly, the cartesan product of n copies of A is denoted as A^n.

For example: $\{0, 1\}^n$ denotes the set of all binary strings of length n.

Observation 2.1 *The cardinality of the set $A \times B$ is the product of the cardinalities of A and $|B|$.*

$$|A \times B| = |A| \times |B|$$

So by the same observation, $|A^n| = |A|^n$

5. **Power set** Given a set A the power-set of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.

   Observation 2.2 *The cardinality of the set $\mathcal{P}(A)$ is $2^{|A|}$, because every element can either appear or not appear.*

   Remark 2.3 *The empty set or the set with no element is either represrented as $\{\}$ or as $\emptyset$.*

2.2 Pictorial representation of Sets

The universe set is usually pictorially represented by a rectangle. The sets are usually represented by circles inside the rectangle with the meaning that the area inside the circle is in the set and the area outside the set but inside the rectagle is the complement of the set. This way of representing sets is called **Venn Diagram** after John Venn whose conceived this idea in 1880.

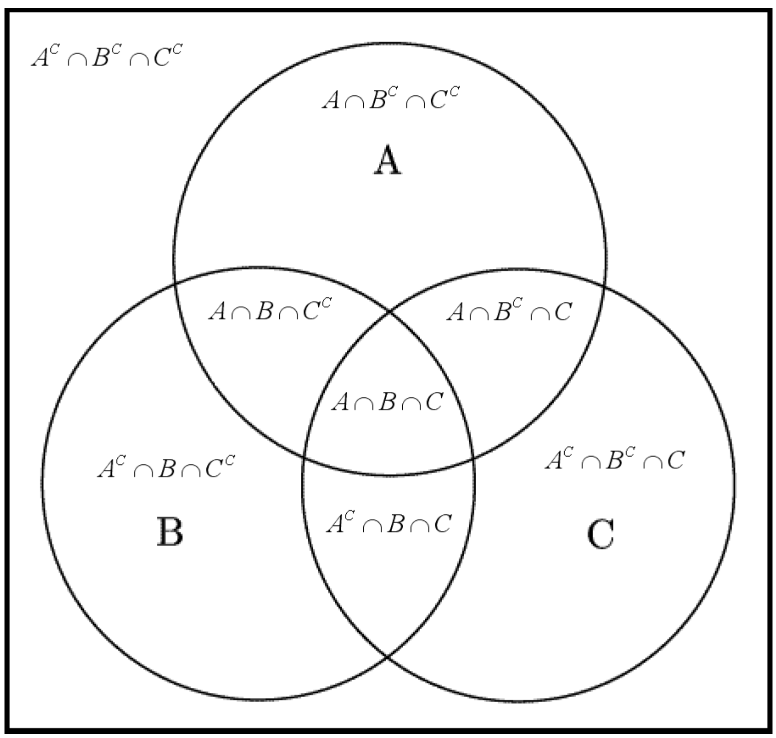

Figure 1: Venn Diagram

2.3 Rules

There are various laws that the set operations follows. The proofs of these law can be obtained by following the venn-diagram. But it is a good exercise to write down the formal proof of these laws.

1. **Associative law of intersection and union** For any sets A, B and C the following holds
   - $((A \cap B) \cap C) = (A \cap (B \cap C))$
   - $((A \cup B) \cup C) = (A \cup (B \cup C))$
2. **Commutative rule of intersection and union** For any sets A and B the following holds
   - $(A \cap B) = (B \cap A)$
   - $(A \cup B) = (B \cup A)$
3. **Distribution rule of intersection and union** For any sets A, B and C the following holds
   - $(A \cup (B \cap C)) = (A \cup B) \cap (A \cup C)$
   - $(A \cap (B \cup C)) = (A \cap B) \cup (A \cap C)$
4. **De Morgan's Law** For any sets A and B the following holds
   - $(A \cap B)^c = (A^c \cup B^c)$
   - $(A \cup B)^c = (A^c \cap B^c)$

As an example of how to prove the set theoretic iequalities/inequalities we present the proof of one the distribution rule.

Theorem 2.4 *For all sets A, B, C we have $(A \cup (B \cap C)) = (A \cup B) \cap (A \cup C)$.*

Proof. We will show the equality using two steps. In Step 1 we will show that for any element $x \in (A \cup (B \cap C))$ we have $x \in (A \cup B) \cap (A \cup C)$. This would imply the $(A \cup (B \cap C)) \subseteq (A \cup B) \cap (A \cup C)$. In Step 2 (using similar arguments as Step 1) we will show that for any $y \in (A \cup B) \cap (A \cup C)$ we have $x \in (A \cup (B \cap C))$. This would imply $(A \cup B) \cap (A \cup C) \subseteq (A \cup (B \cap C))$. Now if for any two sets S and T we have $S \subseteq T$ and $T \subseteq S$ then we have $S = T$. Thus if we prove Step 1 and Step 2 we will have the theorem. In this write-up we will will just give the arguments for Step 1 and we leave formal proof of Step 2 for the reader to complete.

So let $x \in (A \cup (B \cap C))$. Now we can have two disjoint cases. Case 1 is $x \in A$ and Case 2 is $x \notin A$. We will deal will both the cases separately.

Case 1 ($x \in A$): In this case since x is in A so $x \in (A \cup B)$ also $x \in (A \cup C)$. So $x \in (A \cup B) \cap (A \cup C)$.

Case 2 ($x \notin A$): In this case since $x \in (A \cup (B \cap C))$ and $x \notin A$, so $x \in (B \cap C)$. This means that $x \in B$ and $x \in C$. So $x \in (A \cup B)$ and $x \in (A \cup C)$ and hence we have $x \in (A \cup B) \cap (A \cup C)$.

□

3 Relations

Given sets A and B we are often interested is understanding how the elements of the sets interact with each other. Formally it is called a binary relation. A relation is a subset of $S \times T$. So if R is a relation between elements of S and elements of T then $R \subseteq S \times T$. And if $a \in A$ and $b \in B$ are such that $(a, b) \in R$ then we say "a is related to b" and it is denoted by aRb or $a \sim b$ or just by $(a, b) \in R$.

Many times we are interested in relations of elements in a single set S. In that case the relation is a subset of $S \times S$.

A relation can have many kinds of properties. Some of the properties that are commonly studied are presented below:

1. **Symmetric Relation** R is called a symmetric relation if for all elemnent a, b if $(a, b) \in R$ then $(b, a) \in R$.

2. **Reflexive Relation** R is called a reflexsive relation if for all elements in the set is question $(a, a) \in R$ that is a is related to itself.

3. **Transitive Relation** R is called a symmetric relation if for all elemnent a, b, c if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

4. **Anti-Symmetric Relation** R is called an anti- symmetric relation if for all elemnent a, b if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

5. **Connex Relation** R is called a connex relation if for all elements a, b in the set is question either $(a, b) \in R$ or $(b, a) \in R$ or both.

One can also define more general relations (other than binary relations) by defining relations between more that two elements.

3.1 Representation of Binary Relations as graphs

Binary relations are one of the most well studied subjects in the computer science. In fact we have a very nice pictorial way of representing binary relations. We represents binary relations using **Graphs**. A graph G comprises of a set of elements, usually denoted V, called the **vertices** (plural for vertex) and a set of relations, usually denoted E, called the **edges**. So the graph G is written as (V, E) where $E \subseteq V \times V$.

Pictorially, we draw vertices (also called nodes) as balls and edges (also called arcs) as arrows. So if v, w are two vertices and v is related to w, that is $(v, w) \in E$ then we represent it by drawing an arrow from the ball representing v to the ball representing w.

The following is an example of a graph, where the set of vertices is $\{A, B, C, D, E, F\}$ and the set of relations is $\{(A, B), (B, C), (C, E), (E, D), (E, F), (D, B)\}$.

If the relation is symmetric then instead of drawing arrows from a to b and b to a we represent the relation by drawing a line between a and b. This is called **undirected graph**, while if the relation is not symmentric then the graph is called **directed graph**.

If the relation is reflexive then instead of putting arrows from every vertex to itself (we call them **self-loops**) we don't put any self-loops. An undirected graph with no self loops is called a **simple graph**.

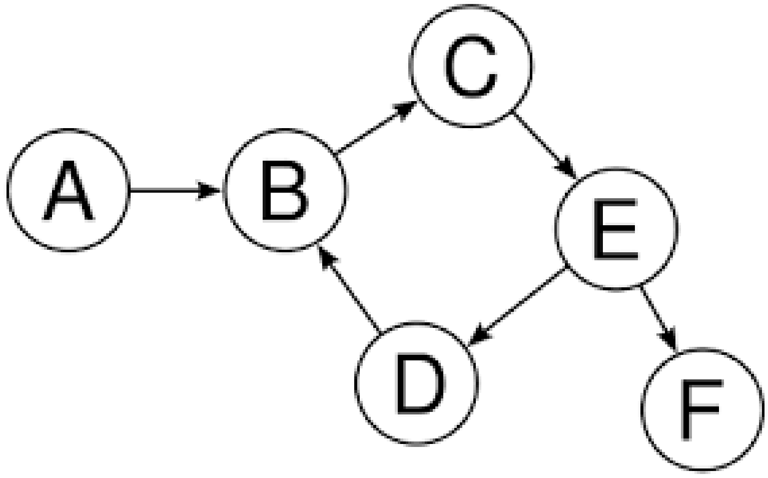

Figure 2: Graph

If the relation is a multiset then the graph representing the relation can have multiple edges between two vertices.