Discrete Mathematics 24/07/19

Lecture 1: Relations, Ordering and Functions

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1 Relation and Graphs

1.1 Introduction

Graphs are nothing but a way of visualizing relations. For an example,

Let S be a set of students in a class.

$R \rightarrow uRv$[1] if u and v are friends on facebook.

i.e, $R \subseteq S \times S\{(u, v) \mid \text{u and v are friends in FB}\}$

The relation can also be represented using graph **??**. According to the graph, $(v_1, v_2) \in R$ while $(v_4, v_5) \notin R$.

In this way, graphs are used to visualize relations. Let us consider the claim that any two persons in this world are connected by six friendship paths. The graphical representation of that is as follows.

Let G(V,E) denote the corresponding universal friendship graph. Then there will be a path of length at most 6 between any two vertices in the graph.

This friendship relation is symmetric in nature. *i.e*, $uRv \implies vRu$. A symmetric relation implies that the edges are undirected in the corresponding graph.

Similarly, the graphical representation of a reflexive relation will have self loops for all the vertices. However, it may be noted that if there is a self loop for all the vertices, then it becomes superfluous and thus can be ignored altogether.

A graph that is undirected and does not have any self loops is called a **simple graph**.

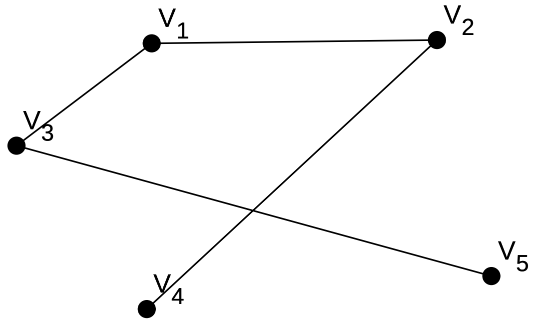

Figure 1: Friendship graph

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[1] The notations uRv and $(u, v) \in R$ are used interchangeably.

1.2 Equivalence relation

If a relation is symmetric, reflexive and transitive, then it is called an **equivalence relation**.

Before exploring equivalence relations, we define partitions.

Definition 1.1 *A partition of a set S is a collection of subsets $P_1, P_2, \ldots, P_t$ s.t,*

1. $\forall i, j \ P_i \cap P_j = \varnothing$
2. $P_1 \cup P_2, \cdots \cup P_t = S$
3. $P_i \neq \varnothing \ \forall i$

Theorem 1.2 *If R is an equivalence relation on S then R partitions S.*

Proof. We start by defining subsets P_i for every $i \in S$ as $P_i = \{x \mid (i, x) \in R\}$. P_i is called equivalence class of i. The theorem statement essentially means that the equivalence classes of R form a partition of S.

To prove this, we prove the following points.

1. If $(a, b) \in R$ then $P_a = P_b$.

   Suppose, $x \in P_a$

   $\implies (x, a) \in R$

   $\because aRb \implies bRa$

   $\therefore (x, a) \in R$

   $\therefore P_a \subseteq P_b$

   Similarly, $x \in P_b \implies x \in P_a$

   $\therefore P_b \subseteq P_a$

   $\implies P_a = P_b$

2. If $(a, c) \notin R$ then $P_a \cap P_c = \emptyset$

   proof by contradiction:

   Let's say $y \in (P_a \cap P_c)$

   $\implies y \in P_a$

   $\implies (y, a) \in R \implies (a, y) \in R$

   Also, $y \in P_c \implies (y, c) \in R$

   $\therefore (a, c) \in R$, *a contradiction.*

3. $\forall x \in S$, $x \in P_x$

   This is because of the reflexive property of R.

4. $\cup_{x \in S} P_x = S$

   This follows from 3.

□

The subsets P_i are called equivalence classes of R

1.3 Problem

In a simple graph, for $u, v \in V$, u is called to be connected to v , if $\exists\ u_o = u, u_1, \ldots, u_k = v$ s.t $(u_i, u_{i+1}) \in E\ \forall i = \{0, 1, \ldots, k-1\}$. Define this as an equivalence relation.

2 Ordering

If R is a relation that is transitive, reflexive, and anti-symmetric, then R is called an ordering.

The anti symmetric property is defined as follows:

1. If $(u, v) \in R$ then $(v, u) \notin R$
2. If $(u, v) \in R$ and $(v, u) \in R$ then $u = v$

Orderings are visualized using **hasse diagrams**, as shown in **??** .

Let A,$B \subseteq P(I_n)$ for some $n \in N$. The following hasse diagram visualizes the ordering $B \preceq A$ if $A \subseteq B$

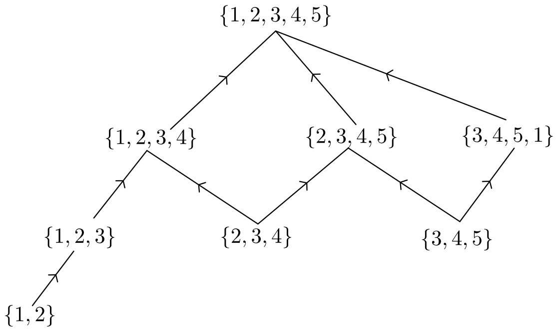

Figure 2: Part of the hasse diagram for the ordering

Theorem 2.1 *If $\preceq$ is an ordering over a finite set S, then*

1. $\exists x \in S$ *s.t* $\nexists y \in S$ *s.t* $y \preceq x$.

   In this case x is called a minimal element.

2. $\exists x \in S$ *s.t* $\nexists y \in S$ *s.t* $x \preceq y$.

   In this case x is called a maximal element.

A set S is called total ordered w.r.t$\preceq$ if $\forall x, y \in S$ either $x \preceq y$ or $y \preceq x$.

If there are one or more incomparable pairs in S w.r.t an ordering $\preceq$, then it is called a partially ordered set.

3 Function

Definition 3.1 *A function f from S to T is a relation between S and T with a property $f \subseteq S \times T$ s.t $\forall x \in S \exists y \in T$ s.t $(x, y) \in f$.*

3.1 Problem

How to compare cardinality of two sets(say S and T)?

To show that $|S| = |T|$ we can show that

1. $|S| \leq |T|$

   If f is one to one, that is $x \neq y \implies f(x) \neq f(y)$, then $|S| \leq |T|$.

2. $|S| \geq |T|$

   If f is onto,

   That is, $\forall y \in T$, $\exists\ x \in S$ s.t $f(x) = y$, then $|T| \leq |S|$.

This concept also helps us in understanding cardinality of infinite sets, as cardinality can not be defined for infinite sets, but can only be compared.

Theorem 3.2 *Cardinality of set of all natural numbers (N) and set of all integers (Z) is the same.*

Proof.

1. $|N| \leq |Z|$

   We can form a one to one function $f : N \rightarrow Z | f(x) = x$.

2. $|Z| \leq |N|$

   In this case, we form a function $g : Z \rightarrow N\ | g(x) = 2x$ if $x \geq 0$ and $g(x) = 2x + 1$ if $x < 0$.

   Now, g is also one to one, and hence $|Z| = |N|$

□

Theorem 3.3 *Cardinality of the set of all natural numbers (N) and set of all rational numbers (Q) is the same.*

Proof.

1. $|N| \leq |Q|$

   As shown in the previous case, we can form a one to one function $f : N \rightarrow Z | f(x) = x$.

2. $|Q| \leq |N|$

   We form the following one to one function to complete our proof:

   We represent any rational number $q \in Q = \frac{q_a}{q_b}$ where $gcd(a, b) = 1$

   We form the function $g : Q \to N \; |g(q) = g(\frac{q_a}{q_b}) = 2^{q_a}3^{q_b}$.

   As the function g is also one to one, we can conclude that $|Q| = |N|$

□

It may be noted that we can also show that the cardinality of a set S is equal to that of n by forming an enumeration of that set's elements.

An enumeration is a surjective mapping of N to S, which is written as follows:

We take variables $x_i\{i \in N\}$ and assign elements of set S to variables x_i uniquely and exhaustively. That is,

1. if $x_i = x_j$ then $i = j$.
2. $\forall s \in S \; \exists \; i \in N$ s.t $x_i = s$.

   This can thus

Definition 3.4 ***A set that has the same cadinality as some subset of*** N ***is called a countable set.***

Theorem 3.5 *The cardinality of the set of all real numbers* R *is more than the cardinality of set of all natural numbers.*

Proof.

Proof by contradiction:

Let $|R| = |N|$.

Then there exists an enumeration of R. Let that be $X\{x_i \; \forall i \in N\}$.

We write the real numbers in their decimal form. Suppose the enumeration looks like as follows:

$x_1 = 0.12345\ldots$

$x_2 = 0.23145\ldots$

$\ldots$

and so on.

We can always form a new real number say r_{new} such that the first digit after decimal is different than that of the first digit after decimal of x_1.

The second digit after the decimal is different than that of the second digit after decimal of x_2, and so on.

By repeating this process for all x_i we obtain r_{new} s.t $\nexists x_i$ so that $x_i = r_{new}$, a contradiction.

This method is called the Canter's diagonalization method.

□