Discrete Mathematics 02/09/2019

Lecture 11: Counting using Generating Functions.

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1 Solved Problems

1.1 Example 1 : How many integer solution to the equation $a + b + c = 6$ for $1 \leq b, c \leq 4$ and $-1 \leq a \leq 2$?

Solution: We can write the generating function for given equation as follows,

$$(x^{-1} + x^0 + x^1 + x^2)(x^1 + x^2 + x^3 + x^4)(x^1 + x^2 + x^3 + x^4)$$

$$= \frac{1}{x^2}(x^1 + x^2 + x^3 + x^4)^3$$

$$= \frac{x^3}{x^2}(1 + x^1 + x^2 + x^3)^3$$

$$= x(1 + x^1 + x^2 + x^3)^3$$

So, required result is coefficient of x^5 in $(1 + x^1 + x^2 + x^3)^3$

coefficient of $x^5 = 12$

So, there are 12 integer solution for the given equation.

1.2 Example 1 : How many ways one can pay 100 Rs using 1 Rs, 2 Rs, 5 Rs, 10 Rs and 50 Rs notes.

Solution: We can write the generating function for given equation as follows,

$$(1 + x^1 + x^2 ... + x^{100})(1 + x^2 + x^4 ... + x^{100})(1 + x^5 + x^{10} ... + x^{100})(1 + x^{10} + x^{20} ... + x^{100})(1 + x^{50} + x^{100})$$

we can extend the generating function up to infinite

$$= (1 + x^1 + x^2 + x^3 ... + x^{100} ...)(1 + x^2 + x^4 + x^6 ... + x^{100} ....)(1 + x^5 + x^{10} ... + x^{100} ...)(1 + x^{10} + x^{20} ... + x^{100} ...)(1 + x^{50} + x^{100})$$

$$= \frac{1}{1-x} \; \frac{1}{1-x^2} \; \frac{1}{1-x^5} \; \frac{1}{1-x^{10}} \; (1 + x^{50} + x^{100})$$

$= \frac{1+x}{1-x^2} \ \frac{1}{1-x^2} \ \frac{1+x^5}{1-x^{10}} \ \frac{1}{1-x^{10}} \ (1 + x^{50} + x^{100})$

$= \frac{1+x}{(1-x^2)^2} \ \frac{1+x^5}{(1-x^{10})^2} \ (1 + x^{50} + x^{100})$

$= (1+x) \ (1+x^5) \ \frac{(1+x^2++x^4+x^6+x^8)^2}{(1-x^{10})^4} \ (1 + x^{50} + x^{100})$

$= (1+x) \ (1+x^5) \ (1+x^2++x^4+x^6+x^8)^2 \ (1-x^{10})^{-4} \ (1 + x^{50} + x^{100})$

So, required result is coefficient of x^{100} in above expansion.

2 Principle of Inclusion and Exclusion

2.1 What is Inclusion and Exclusion

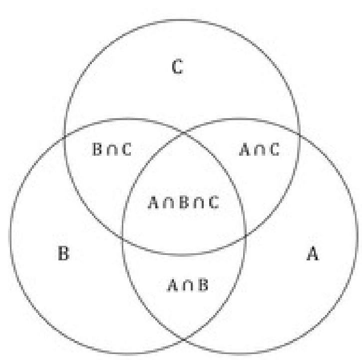
Solution: In combinatorics, the inclusion-exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets
symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where A and B are two finite sets and $|S|$ indicates the cardinality of a set S. The formula expresses that the sum of $|A|$ and $|B|$ may be greater than actual value since some elements may be counted twice. The double-counted elements are nothing but the intersection elements of the sets and the count is corrected by subtracting the size of the intersection.
The principle is more clearly seen in the case of three sets, which for the sets A, B and C is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

This formula can be verified by counting how many times each region is included in the Venn diagram. In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.



Generalizing the results of these examples gives the principle of inclusionexclusion.
To find the cardinality of the union of n sets:

1. Include the cardinalities of the sets.
2. Exclude the cardinalities of the pairwise intersections.
3. Include the cardinalities of the triple-wise intersections.
4. Exclude the cardinalities of the quadruple-wise intersections.
5. Include the cardinalities of the quintuple-wise intersections.
6. Continue, until the cardinality of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).

In its general form, the principle of inclusion-exclusion states that for finite sets $A_1, \ldots, A_n$, one has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

This can be compactly written as

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}| \right)$$

or

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{S \subseteq \{1,2,3\ldots n\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

2.2 Proof

For finite set $A_1, A_2, A_3 \ldots A_n$

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{S \subseteq \{1,2,3\ldots n\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

We can prove this by induction on n. For n=1, it is trivial:

$$|A_1| = \sum_{S \subseteq \{1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

For inductive step, we will take it as given that:

$$|A_1 \cup A_2 \cup \ldots \cup A_n - 1| = \sum_{S \subseteq \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

and thereby find $|A_1 \cup A_2 \cup \ldots \cup A_n|$:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |(A_1 \cup A_2 \cup \ldots \cup A_n - 1) \cup A_n|$$

$$= |A_1 \cup A_2 \cup \ldots \cup A_n - 1| \cup +|A_n| - |(A_1 \cup A_2 \cup \ldots \cup A_n - 1) \cap A_n|$$

$$= |A_1 \cup A_2 \cup \ldots \cup A_n - 1| \cup +|A_n| - |(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_n - 1 \cap A_n)|$$

$$= \sum_{S \subseteq \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right| + |A_n| - \sum_{S \subset \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} | \bigcap_{j \in S} A_j \cap A_n|$$

$$= \sum_{S \subseteq \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right| + |A_n| - \sum_{S \subset \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \left(\bigcap_{j \in S} A_j \right) \cap A_n \right|$$

$$= \sum_{S \subseteq \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right| + |A_n| - \sum_{S \subset \{1,2,3 \ldots n-1\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in (S \cup \{n\})} A_j \right|$$

$$= \sum_{S \subseteq \{1,2,3 \ldots n\}; S \neq \phi} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

2.3 References

1. https://en.wikipedia.org/wiki/Inclusion
2. http://aleph.math.louisville.edu/teaching/2009FA-681/notes-090901.pdf