

Lecture 12: More on Distribution of Balls into Bins

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1 Using Generating Function

In this section we will see how to use generating function instead of combinatorial method to solve the problem.

N Balls, K Bins

1.1 Distinct Bin (Empty Allowed)

Here each k bins can have 0 to n balls, for generality we are taking it to infinity.

Indistinguishable ball

$$\underbrace{(1 + x + x^2 + \dots)(1 + x + x^2 + \dots) \dots (1 + x + x^2 + \dots)}_{k\text{-times}}$$

$$= \frac{1}{(1-x)^k} = (1-x)^{-k}$$

$$\text{Coefficient of } x^n = (-1)^n \binom{-k}{n} = \frac{(-1)^n (-k)(-k-1)\dots(-k-n+1)}{n!}$$

$$= \frac{(k)(k+1)\dots(k+n-1)}{n!}$$

$$= \binom{n+k-1}{k-1}$$

Distinguishable ball, Ordered Balls

N balls can be arranged in $n!$ ways.

$$n! \binom{n+k-1}{k-1}$$

Distinguishable ball, Non-ordered Balls

k^n (Try using generating function)

1.2 Distinct Bin (Empty Not Allowed)

Here each k bins can have 1 to n balls, for generality we are taking it to infinity.

Indistinguishable ball

$$\underbrace{(x + x^2 + x^3 \dots)(x + x^2 + x^3 \dots) \dots (x + x^2 + x^3 + \dots)}_{k\text{-times}}$$

$$= \frac{x^k}{(1-x)^k} = x^k(1-x)^{-k}$$

Coefficient of x^n (coefficient of x^{n-k} in $(1-x)^{-k}$)

$$= (-1)^n \binom{-k}{n-k}$$

$$= \binom{n-1}{k-1}$$

Distinguishable ball, Ordered balls

N balls can be arranged in $n!$ ways.

$$n! \binom{n-1}{k-1}$$

Distinguishable ball, Non-ordered balls

$$n! \underbrace{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)}_{k\text{-times}}$$

$$= n!(e^x - 1)^k = n!(-1)^k(1 - e^x)^k$$

$$= n!(-1)^k \left(1 - ke^x + \binom{k}{2}e^{2x} - \binom{k}{3}e^{3x} \dots\right)$$

$$\text{Coefficient of } x^n = n!(-1)^k \sum_{i=0}^k (-1)^i \binom{k}{i} \frac{i^n}{n!} = \sum_{i=0}^k (-1)^{k+i} \binom{k}{i} i^n$$

Now using mutual inclusion and exclusion for same

A_i = Set of distribution such that bin i is empty.

$$k^n - \left| \bigcup_{i=1}^n A_i \right|$$

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|.$$

$$= \binom{k}{1}(k-1)^n + \binom{k}{3}(k-3)^n + \dots + (-1)^{i+1} \binom{k}{1}(k-i)^n$$

$$k^n - \left| \bigcup_{i=1}^n A_i \right| = \sum_{i=0}^k (-1)^i \binom{k}{k-i} (k-i)^n$$

2 Non-Distinct bins

N balls → K bins ↓	Indistinguishable	Distinguishable (Ordered)	Distinguishable (Unordered)
Empty allowed	$P(n, k)$	$n!P(n, k)$	$\sum_{i=1}^k S_2(n, i)$
Empty not allowed	$P(n, k) - P(n, k - 1) = S_1(n, k)$	$n!S_1(n, k)$	$S_2(n, k)$

3 Exercise

3.1 Problem 1

Given a graph G and n vertices and m edges. Show that there exist a subgraph H of G that is bipartite and has at least $\frac{m}{2}$ edges.

3.2 Problem 2

A city of n people want to form a club of people with given condition

- Each club must have odd number of people.
- Intersection of two club is even.

What is the maximum number of club you can make ?

3.3 Problem 3

Prove: $\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$