Discrete Mathematics 18/09/2019

Topic: How recurrences can be used to solve problems?

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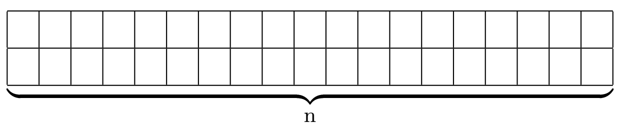
1 Problems on Recurrence

1.1 Problem-1: There is a room of size 2xn. Find the #of possible tilings of 2xn using the tile size 1x2.

Solution: Let T(n)= # of possible tiling of 2Xn
If $n = 1$, then tiling can be done only 1 way.
If $n = 2$, there are two ways to do tiling, first horizontally, and second vertically.



Now for n, if we place one 1x2 tile vertically, then the problem reduce to the size of n-1, and if we place one 1x2 tile horizontally, then we have to cover 2x2 size tile and problem will reduced to the size of n-2.

So, with base case T(1) = 1, T(2) = 2

$$\boxed{T(n) = T(n-1) + T(n-2)} \tag{1}$$

1.2 Problem-2: How many ways, n digit numbers are there such that no two consecutive digits are same.

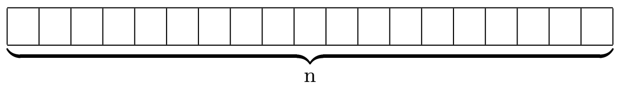
Solution: Let T(n) = # of ways such that no two consecutive digits are same

On first place, we can place any one from these 9 choices 1,2,3,4,5,6,7,8,9 and for remaining place there is 9 choices from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 which will different from their consecutive digits.

$$\boxed{T(n) = 9 \times 9 \times 9 \times 9 \times 9 ...... \times 9 = 9^n}, \tag{2}$$

1.3 Problem-3 How many ways, n digit numbers are there such that no two consecutive digits are 8.

Solution: Let T(n) = # of ways such that no two consecutive digits are 8.



There are 2 cases:
Case 1: If the last $digit \neq 8$, this an be done in $9 * T(n-1)$ ways (because we can put any digit except 8).
Case 2: If the last digit = 8 and second last $\neq 8$, this can be done in $9 * T(n-2)$ (we can put any digit at second last position except 8)

So, with base cases T(1) = 9, T(2) = 89

$$\boxed{T(n) = 9T(n-1) + 9T(n-2)} \tag{3}$$

1.4 Problem-4 How many ways, n digit numbers are there such that no three consecutive digits are 8.

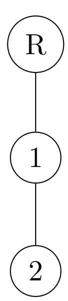
Solution: $[Homework]$

2 Recurrence on graphs

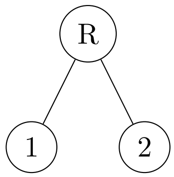
2.1 Problem-1 How many n vertex rooted trees are there?

Solution: G_1 and G_2 are isomorphic to each-other, if:
$\sigma : V_{G_1} \rightarrow V_{G_2}, (i,j) \in E(G_1) \Leftrightarrow (\sigma(i), \sigma(j)) \in E(G_2)$

For no. of nodes= 3:

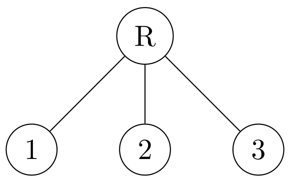


Code for this tree = 0011 { Here R represents the root of the tree.}

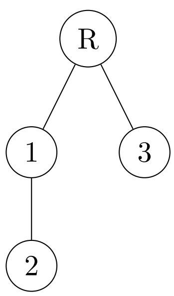


Code for this tree = 0101

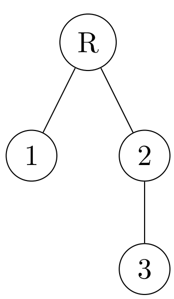
For no. of nodes= 4:



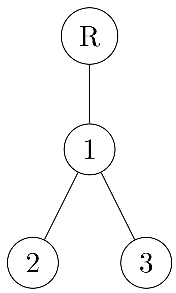
Code for this tree = 010101



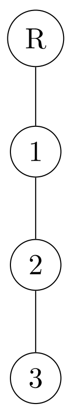
Code for this tree = 001101



Code for this tree = 010011



Code for this tree = 001011



Code for this tree = 000111

Length of Strings = $2(n-1)$, start with 0 and end with 1. In the first i position, $\#0 \geq \#1$

2.2 Read Graph Isomorphism Algorithm

No. of mapping from n points to n points = $n!$

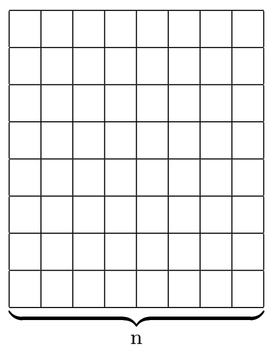
Quasi-polynomial: $n! \to 2^{O(n)} \to 2^{O(\log n)^8} \to 2^{(O \log n)} = n^c$

Exercise-1 Find a non-planar graphs.
Exercise-2 Prove that trees are planar.
Exercise-3 How many n vertex rooted trees are there.

3 Catalan Number

$\#of\{0,1\}^{2n}$ string set
#0 = n, #1 = n.
In the first k position $(\forall k)$, $\#0 \geq \#1$

No. of ways to go from $(0,0)$ to (n,n), $C_n = 2n_{C_n}$
No. of ways to go from $(0,0)$ to (n,n) such that no. of horizontal moves is always greater than # of vertical moves:



#Paths = $\bigcup_{i=1}^{n} A_i$ {Number of paths that touch the diagonal at (i,i) for the first time = $P_n(i)$ };

$P_n(1)$, $P_n(2)$, $P_n(3)$, ..., $P_n(i)$, ..., $P_n(n)$

$P_n(i) = C_{i-1}C_{n-i}$
$P_n(n) = C_{n-1}C_0$
$C_0 = 1$ [defined]

With base case: $C_1 = 1, C_2 = 2$

$$\boxed{C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}} \tag{4}$$

Exercise 4 Prove that, $D_n = (n-1)D_{n-1} + (-1)^n$
Exercise 5 Read wiki page of Catalan Number.

4 Types of Recurrences:

4.1 Linear Recurrence

$T(n) = aT(n-1) + bT(n-2) + cT(n-3) + d$, with $T(1), T(2), T(3)$

4.2 Non-linear Recurrence

$C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i}$

4.3 Exponential Recurrence

$D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$