

Lecture 4: Recurrence Relation using Generating functions

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1 How to solve Recurrence relation using generating functions.

For the sequence $a_0, a_1, a_2, a_3, a_4, \dots, a_n$ the generating functions can be written as-

$$P(x) = \sum_{i=0}^n a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n$$

OR

$$P(x) = \sum_{i=0}^n \frac{a_i}{n!} x^i$$

So in a general form we can write a generating function as-

$$P(x) = \sum_{i=0}^n f(a_i) x^i$$

2 General View

Given a recurrence relation for the sequence (a_n) , we

1. Deduce from it, an equation satisfied by the generating function $a(x) = \sum_{i=0}^n a_i x^i$.
2. Solve this equation to get an explicit expression for the generating function.
3. Extract the coefficient a_n of x_n from $a(x)$, by expanding $a(x)$ as a power series.

3 Some Examples

1. **Fibonacci Series:** Recurrence function given as:

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \quad F_1 = 1 \quad F_0 = 0$$

2. **Catalan's Number** Recurrence function given below:

$$C_n = \sum_{i=1}^n C_i C_{n-i} \quad C_1 = 1 \quad C_2 = 2$$

3. **Dearrangement** :Recurrence function given as:

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

Problem 1 Solve the below recurrence relations using generating function:- $F_n = F_{n-1} + F_{n-2}$ $n \geq 2$ $F_1 = 1$ $F_0 = 0$

Solution 1 Let $P(x)$ be the generating function given as-

$$P(x) = \sum_{i=0}^n F(i)x^i \text{ where } F(i) \text{ is the fibonacci no.}$$

$$\implies 0 + x + x^2 + 2x^3 + 3x^4 + \dots$$

$P(X)$ can also be written as:

$$P(X) = X + \sum_{i=2}^n F(i)x^i$$

$$\implies X + \sum_{i=2}^n (F(i-1) + F(i-2))x^i$$

$$\implies P(X) = X + \sum_{i=2}^n F(i-1)x^i + \sum_{i=2}^n F(i-2)x^i$$

$$\implies P(X) = X + X \sum_{i=2}^{n-1} F(i-1)x^i + X^2 \sum_{i=2}^{n-2} F(i-2)x^i$$

$$\implies P(X) = X + XP(X) + X^2P(X)$$

$$\therefore (1 - X - X^2)P(X) = X$$

$$P(X) = \frac{X}{(1 - X - X^2)}$$

$$P(X) = \frac{-X}{(X^2 + X - 1)}$$

$$\therefore P(X) = \frac{\alpha}{X - \phi} + \frac{\beta}{X - \gamma}$$

$$\phi\gamma = -1 \quad \phi - \gamma = -1$$

$$\phi = \frac{-1 - \sqrt{5}}{2} \quad \gamma = \frac{-1 + \sqrt{5}}{2}$$

$$\alpha + \beta = 1 \quad \alpha\gamma + \beta\phi = 0$$

$$\alpha = \frac{-1 - \sqrt{5}}{2\sqrt{5}} \quad \beta = \frac{-1 + \sqrt{5}}{2\sqrt{5}}$$

$$P(X) = \frac{\alpha}{-\phi} \frac{1}{(1 + X\gamma)} + \frac{\beta}{-\gamma} \frac{1}{(1 + X\phi)}$$

$$\text{We know, } \frac{1}{(1 + X\phi)} = \sum_{i=0}^{\infty} (-1)^i (X\phi)^i$$

$$\text{and, } \frac{1}{(1 + X\gamma)} = \sum_{i=0}^{\infty} (-1)^i (X\gamma)^i$$

$$\therefore P(X) = \frac{-1}{\sqrt{5}} \sum_{i=0}^{\infty} (-1)^i (X\gamma)^i + \frac{-1}{\sqrt{5}} \sum_{i=0}^{\infty} (-1)^i (X\phi)^i$$

$$\implies P(X) = \sum_{i=0}^{\infty} \left(\frac{-1}{\sqrt{5}} (-1)^i (\gamma)^i + \frac{-1}{\sqrt{5}} \cdot (-1)^i (\phi)^i \right) (X)^i$$

$$\therefore F_n = \frac{-1}{\sqrt{5}} (-1)^n (\gamma)^n + \frac{-1}{\sqrt{5}} \cdot (-1)^n (\phi)^n$$

$$\implies F_n = (-1)^n \cdot \frac{\phi^n - \gamma^n}{\sqrt{5}}$$

$$\implies F_n = \frac{\phi^n - \gamma^n}{\phi - \gamma}$$

Problem 2 Solve the below recurrence relations using generating function-:(For exercise)

$$F_n = F_{n-1} + F_{n-2} + F_{n-3} + 2 \quad n \geq 2 \quad F_1 = F_0 = 0 \quad F_1 = 1$$