Discrete Mathematics 25/09/2019

Lecture 15: Solving recurrences of Catalan Number and Derangement

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1 Solving recurrences for derangement

A derangement is a permutation in which none of the elements is mapped to itself. The formula for the number of derangements D_n on n elements is given by -

$$\mathrm{D}_n = n! \textstyle\sum_{k=0}^{n} \dfrac{(-1)^k}{k!}$$

The recursive formula for derangement on n objects is thus given by

$$\mathrm{D}_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

$$\Rightarrow D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2})$$

$\Rightarrow A_n = (-1)A_{n-1};$
$where (A_n = D_n - nD_{n-1})$

$$\Rightarrow A_n = (-1)^n A_1$$

Thus,
$\mathrm{A}_n = (-1)^n A_2 = (-1)^n$

Therefore, $\mathrm{D}_n = nD_{n-1} + (-1)^n$

We use the following exponential generating function to solve this simplified recurrence relation−

$$\mathrm{f(x)} = \textstyle\sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$$

where D_n =n!*coefficient of x^nin f(x)

$$\mathrm{f(x)} = \textstyle\sum_{n=0}^{\infty} D_n \frac{x^n}{n!} = \mathrm{D}_0 + \sum_{n=1}^{\infty} D_n \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} D_n \frac{x^n}{n!}$$

$$\Rightarrow f(x) = 1 + \textstyle\sum_{n=1}^{\infty} (nD_{n-1} + (-1)^n) \frac{x^n}{n!}$$

$$\Rightarrow \mathrm{f(x)} = 1 + \textstyle\sum_{n=1}^{\infty} nD_{n-1} \frac{x^n}{n!} + \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!}$$

$= 1$+x$\sum_{n=1}^{\infty} D_{n-1}\frac{x^{n-1}}{(n-1)!} + e^{-x} - 1$

=x$\sum_{n=0}^{\infty} D_n\frac{x^n}{n!} + e^{-x}$

$= xf(x) + e^{-x}$

$\Rightarrow (1-x)f(x) = e^{-x}$

$\Rightarrow f(x) = \frac{e^{-x}}{1-x}$

The coefficient of x^{n}in f(x) gives the value of $\frac{D_n}{n!}$

$\Rightarrow$D$_n = n!\sum_{i=0}^{n}$(coefficient of x^i in (1-x)$^{-1}$)*(coefficient of x^{n-i} in e^{-x})

$\Rightarrow D_n = n!\sum_{i=1}^{n}\frac{(-1)^i}{i!}$

2 Solving recurrence of Catalan number

We use the ordinary generating function for Catalan number given by-

f(x)=$\sum_{i=0}^{\infty} C_i x^i$

The recurrence formula involving Catalan numbers(as determined earlier) is given by- C$_{n+1}$ =$\sum_{i=0}^{n} C_{n-i}C_i$

From the given recurrence we obtain - C$_1 = C_0C_1$

$\Rightarrow C_0 = 1$

f(x)=$\sum_{i=0}^{\infty} C_i x^i = C_0 + \sum_{i=1}^{\infty} C_i x^i = 1 + \sum_{i=1}^{\infty} C_i x^i$

$\Rightarrow f(x) = 1 + x\sum_{i=0}^{\infty} C_{i+1}x^i$

$\Rightarrow f(x) = 1 + x\sum_{i=0}^{\infty}(\sum_{j=0}^{i} C_{i-j}C_j)x^i$
(substituting the expression for C$_{i+1}$)

$\Rightarrow f(x) = 1 + x\sum_{i=0}^{\infty}(\sum_{j=0}^{i} C_{i-j}C_j)x^i = 1 + x\sum_{i=0}^{\infty} Y_i x_i$

Y$_i = \sum_{j=0}^{i} C_{i-j}C_j$ =coefficient of x^i in f(x)^2

$\Rightarrow f(x) = 1 + xf(x)^2$
,which is a quadratic equation in f(x)

Thus f(x) can take on either of the two functions−

f(x)= $\dfrac{(1-\sqrt{1-4x})}{2x}$

or

f(x)= $\dfrac{(1+\sqrt{1-4x})}{2x}$

But $\lim_{n\to\infty} \dfrac{(1-\sqrt{1-4x})}{2x} = 1$

while $\lim_{n\to\infty} \dfrac{(1+\sqrt{1-4x})}{2x} = \infty$

$Therefore,$ f(x)= $\dfrac{(1-\sqrt{1-4x})}{2x}$

The nth Catalan number is given by the coefficient of x^n in f(x)

f(x)= $\dfrac{1}{2x}(1-\sqrt{1-4x})$

$$\Rightarrow f(x) = \frac{1}{2x}[1-(1+\binom{\frac{1}{2}}{1}(-4x)+\binom{\frac{1}{2}}{2}(-4x)^2+......\binom{\frac{1}{2}}{n}(-4x)^n+\binom{\frac{1}{2}}{n+1}(-4x)^{n+1}+....]$$

$$Thus, C_n = \frac{\binom{\frac{1}{2}}{n+1}(-4)^n}{2}$$

$$\Rightarrow C_n = \frac{1.3.5.......(2n-1).4^n}{2^n(n+1)!}$$

$$\Rightarrow C_n = \frac{1.3.5.......(2n-1).2^n.2^n.n!}{2^n(n+1)!n!} = \frac{(2n)!}{(n+1).(n!)^2}$$

$$\Rightarrow C_n = \frac{\binom{2n}{n}}{n+1}$$

□