Discrete Mathematics 16/10/19

Lecture 18: MORE ON GRAPH THEORY

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1 Cut Spaces and Cycle Spaces

Let T be a spanning tree of the graph G, then every $e \in T$ corresponds to a cut in G.

In G we have

- Cycle Space C
- Cut Space C*

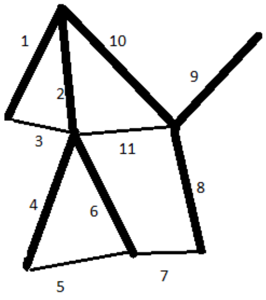

Figure 1: G is the graph and thick lines represent Spanning Tree T

Removing any edge e where $e \in T$ will leave the tree disconnected.It corresponds to a cut i.e. a set of edge in G, removing of those will disconnect the graph and divide the tree in the same way as removing e from T. For example removing 2 from T is equivalent to the cut $\{3, 2, 11, 7\}$ in G.

1. $C = C*^{\perp}$
2. The spanning tree T has edges $e_1, e_2, e_3, ..., e_{n-1}$.
   Every $e \in T$ corresponds to a cut and $A_1^T, A_2^T, A_3^T, ..., A_{n-1}^T$, are the fundamental cuts produced by every $e \in T$. e_i corresponds to the cut A_i^T. Then

$$e_i \in A_i^T$$

$$e_i \notin A_j^T$$

where $i \neq j$

So A_i^T cannot be formed by any linear combination of other A_j^T

$A_1^T, A_2^T, A_3^T, ..., A_{n-1}^T$, are linearly independent.

$\forall e \in G/T$ adding e to the spanning tree T, produces a cycle.

If we add e_i to T we get the cycle B_i^T

So $B_1^T, B_2^T, B_3^T, ..., B_{m-(n-1)}^T$, is the set of fundamental cycles produced by the spanning tree.

$$e_i \in B_i^T$$

$$e_i \notin B_j^T$$

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So B_i^T cannot be formed by any linear combination of other B_j^T

$B_1^T, B_2^T, B_3^T, ..., B_{m-(n-1)}^T$, are linearly independent.

Let $B_1^T, B_2^T, B_3^T, ..., B_{m-(n-1)}^T$, span a subspace $V \subset C$
Then $C^* \subset V^\perp$
If $dim(V) = m - (n-1)$
Then $dim(V^\perp) = (n-1)$
So $dim(C^*) < (n-1)$
which is not possible. So fundamental cycle spans the cycle space.

$C = (C^*)^\perp$
$C \subseteq (C^*)^\perp$
A is a cycle
B is a cut
A.B = 0

The number of common edges in A and B is even as a Cut divides the graph into two disconnected graphs, and there has to be even number of edges common in cut and cycle to end on the same side as started.

2 Jordan Curve Theorem

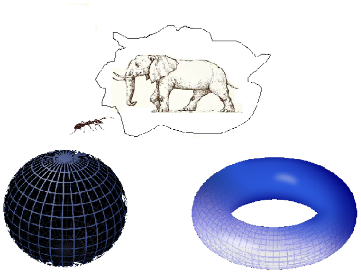


Figure 2: Jordan Curve Theorem

Let us enclose an elephant inside a closed curve. For an ant to bite the elephant, it does not necessarily have to cross the line. It all depends on the type of surface they are on. If they are on a spherical structure like surface of earth, the ant have to cross the line. But if they are on a surface like torus the ant can move around the torus and bite the elephant without crossing the line.
Hence it depends on what surface we consider for it to be planar.

3 Planar Graph

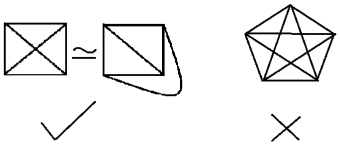


Figure 3: Planar Graph

A graph is called planar graph if there exists a graph isomorphic to the graph drawn in such a way no two edges intersect other than intersecting on their common vertex, if exists any between them.
For example K_4 is a planar graph but K_5 is not.

4 Stereo-graphic Projection

The stereo-graphic projection is a particular mapping (function) that projects a sphere onto a plane in 3-dimensions and a circle onto a line in 2-dimensions. It maps every point except one i.e the top point of the sphere or top point of the circle.

In 2-dimensions a line is drawn from the top point of the circle to the line, crossing the circle. Point in the circle is mapped to the point touched in the line below.

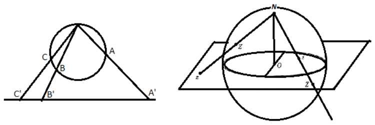
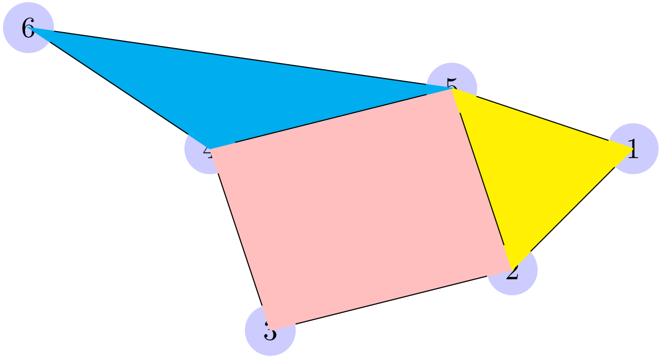


Figure 4: Stereo-Graphic Projections

In 3-dimensions a line in space is subtended touching the sphere intersecting it and touching the plane below it. Point in the sphere is mapped to the point touched in the plane by the line.

Both these mappings are one-one. Points in a circle are mapped to points in a line of infinite length and points in a sphere are mapped to points in a plane of infinite area.

5 Faces



Faces is the region subtended by a set of vertices and by a set of edges bounding them. When a planar graph is drawn in this way, it divides the plane into regions called faces.

6 Euler's Formula

Let n be the number of vertices, m be the number of edges and f be the number of faces or regions in a planar, connected, simple graph G. Then n-m+f=2.

Proof. We prove it by mathematical induction where we induct on f or number of regions or faces.

Base Case:

Let G be a tree. So G does not contain any cycles.The number of regions is one as only outer face is present.

$$f = 1; \tag{1}$$

In a tree the number of edges is one less than that of vertices.

$$m = n - 1; \tag{2}$$

So

$$n - m + f = n - (n - 1) + 1; \tag{3}$$

$$n - m + f = 2; \tag{4}$$

So it is true for f=1

Induction Hypothesis:

Let n-m+f=2 be true for ∀ f ∈ Z^+

Induction Step:

Let G be a planar connected simple graph having k faces i.e. f=k, n1 vertices i.e. n=n1 and m1 edges i.e. m=m1. Let the condition hold for f=k;

$$n1 - m1 + k = 2 \tag{5}$$

If we remove one edge from the graph keeping it connected, then two regions merge into one. f decreases from k to k-1 and number of edges decreases from m1 to m1-1 and n remains same.

$$n1 - (m1 - 1) + (k - 1) = n1 - m1 + k \tag{6}$$

$$n1 - (m1 - 1) + (k - 1) = 2 \tag{7}$$

So it is true for f=k-1. Hence by Principle of Induction n-m+f=2 is true. □