Discrete Mathematics 21 October 2019

Lecture 19: PLANAR GRAPHS

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1 Planar Graph

Definition 1.1 *A graph is called* **Planar** *if* $\exists$ *a drawing on* $\mathbb{R}^2$ *such that no edges intersect.*

Theorem 1.2 (Euler's theorem) *If G=(V,E) is connected planar graph with f faces then*

$$n - m + f = 2$$

where $|V|= n, |E|= m$

Proof. Proof is simple induction on number of edges. □

Theorem 1.3 *For any planar graph G=(V,E), we have* $|E|\leq 3|V|-6$ *where* $|V|\geq 3$

Proof. We know that for any graph G=(V,E)

$$\sum_{i=1}^{|V|} deg(v_i) = 2|E| \qquad \forall v_i \in V$$

Let $f_1, f_2, \ldots, f_{|F|}$ are the faces and $r_1, r_2, \ldots, r_{|F|}$ are number of edges in each face.

$$\sum_{i=1}^{|F|} r_i = 2|E|$$

At least 3 edges are required to construct a face i.e., $r_i \geq 3 \qquad \forall i$

$$3|F|\leq \sum_{i=1}^{|F|} r_i = 2|E|$$

$$\Rightarrow 3|F|\leq 2|E|$$

$$\Rightarrow |F|\leq \frac{2}{3}|E|$$

Now, according to Euler's theorem for any planar graph, we have $|V|-|E|+|F|= 2$

$$\Rightarrow |V|-|E|+\frac{2}{3}|E|\geq |V|-|E|+|F|= 2$$

$$\Rightarrow |V|-\frac{|E|}{3} \geq 2$$

$$\Rightarrow |E|\leq 3|V|-6$$

□

Corollary 1.4 *For planar graphs,*

$$|E| \leq 3|V| - 6 \Rightarrow |E| = \Theta(|V|)$$

So, planar graphs are sparse graphs.

Example 1 *Prove that K_5 is not planar.*

Proof. K_5 has 10 edges and a planar graph with 5 vertices can have at most $3 \cdot 5 - 6 = 9$ edges.
Thus K_5 is not planar □

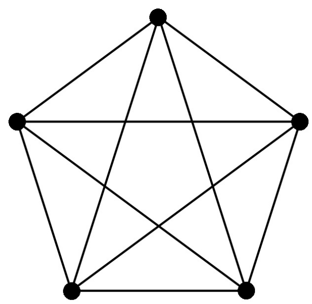

Figure 1: Graphical representation of K_5 [2]

Theorem 1.5 *For any bipartite planar graph G=(V,E), we have $|E| \leq 2|V| - 4$ where $|V| \geq 3$*

Proof. We know that any bipartite graph G=(V,E) has only even length cycles and a face is nothing but a cycle.
Let $f_1, f_2, \ldots, f_{|F|}$ are the faces and $r_1, r_2, \ldots, r_{|F|}$ are number of edges in each face.

$$\sum_{i=1}^{|F|} r_i = 2|E|$$

For bipartite graph, at least 4 edges are required to construct a face i.e., $r_i \geq 4 \quad \forall i$

$$4|F| \leq \sum_{i=1}^{|F|} r_i = 2|E|$$

$$\Rightarrow 4|F| \leq 2|E|$$

$$\Rightarrow |F| \leq \frac{|E|}{2}$$

Now, according to Euler's theorem for any planar graph, we have $|V| - |E| + |F| = 2$

$$\Rightarrow |V| - |E| + \frac{|E|}{2} \geq |V| - |E| + |F| = 2$$

$$\Rightarrow |V| - \frac{|E|}{2} \geq 2$$
$$\Rightarrow |E| \leq 2|V| - 4$$

□

Example 2 *Prove that $K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ has 9 edges and a planar bipartite graph with 6 vertices can have at most $2 \cdot 6 - 4 = 8$ edges.
Thus $K_{3,3}$ is not planar □

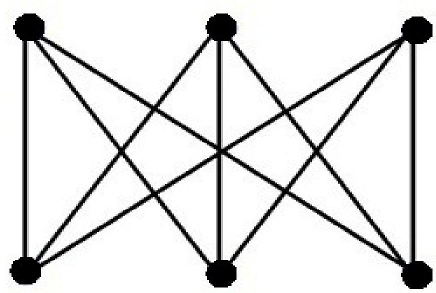

Figure 2: Graphical representation of $K_{3,3}$ [2]

Removing a vertex and its incident edges from a planar graph keeps the graph planar. In other words if G=(V,E) is planar ⇒ G′=G\A and G′′=G\B is also planar $\forall A \subseteq V$, B⊆E. Even merging of two vertices of a planar graph produces another planar graph i.e., if G=(V,E) is planar ⇒ G′′′=G/(u,v) is also planar $\forall (u, v) \in E$

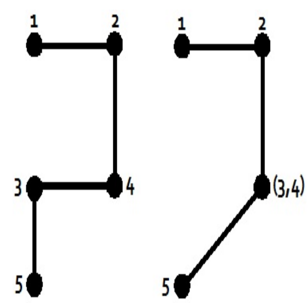

Figure 3: Example of edge shrinking of a graph

Definition 1.6 *H is a* **subgraph** *of G if it can be obtained from G by*

(i) deleting vertices

(ii) deleting edges

Definition 1.7 *H is a* **minor** *of G if it can be obtained from G by*

(i) deleting vertices

(ii) deleting edges

(iii) shrinking edges or merging vertices

If B is minor of A then

(i) A is planar ⇒ B is also planar

(ii) B is non-planar ⇒ A is non-planar

Theorem 1.8 (Kuratowski's theorem) *G has K_5 or $K_{3,3}$ as minor if and only if G is not planar*

2 Graph Coloring

Definition 2.1 Graph Coloring *is an assignment of colors to the vertices of a graph G=(V,E) such that no two adjacent vertices have the same color.*

$$c : V \to \{c_1, c_2, \ldots, c_t\} \text{ such that } \forall (u, v) \in E \Rightarrow c(u) \neq c(v)$$

In simple words, no two vertices of an edge should be of the same color.[1]

Example 3 *Prove that any planar graph can be colored with 6 colors.*

Proof. Proof left as an exercise! □

Reference

1. https://www.tutorialspoint.com/graph_theory/graph_theory_coloring.htm

2. https://miro.medium.com/