

Lecture 20: Kuratowski's Theorem

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1 Review

1.1 Minors

Undirected graph H is a minor of a graph G , if H can be obtained from G by

- Removing Vertices
- Removing Edges
- Shrinking Edges

2 3 - connected Graph

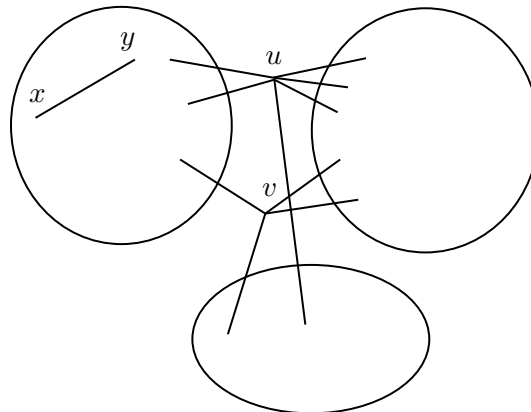
Theorem 2.1 *If G is 3 - connected graph with $|V| \geq 4$, then $\exists (u, v) \in E(G)$ such that the graph G remains 3-connected after shrinking the edge (u, v)*

Proof.

The proof is by Contradiction.

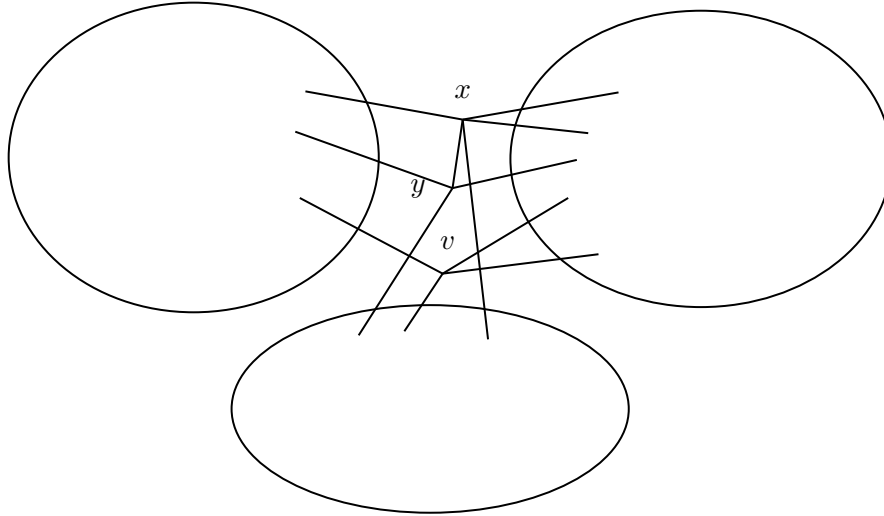
Let's say there does not exist such an edge.

Then, $\forall (x, y) \in E(G)$, the graph G will not be 3 - connected after shrinking the edge (x, y) . i.e $\forall (x, y) \in E(G)$, the graph $G/(x, y)$ consist of two separators u, v such that removing them will make the graph $G/(x, y)$ disconnected.

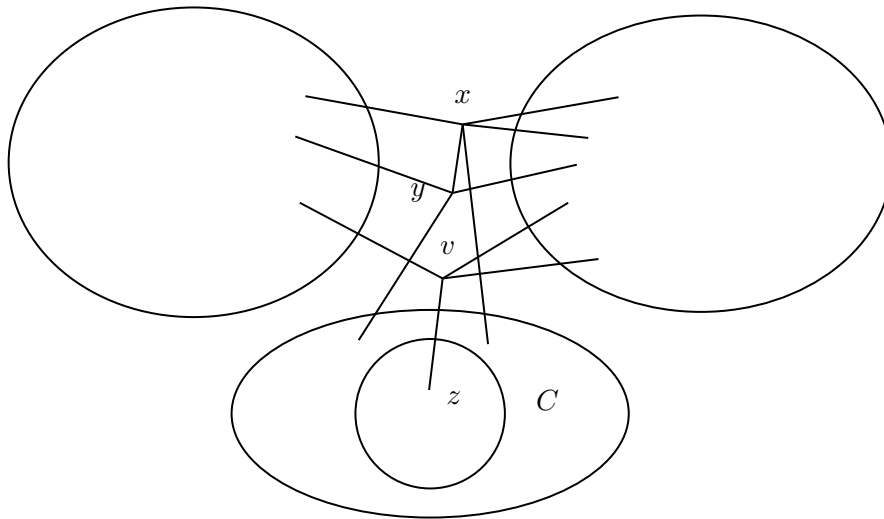


Either u or v must be the vertex corresponding to the shrunk edge (x, y) . If it is not the case, removing u, v from the original graph G will make it disconnected contradicts the

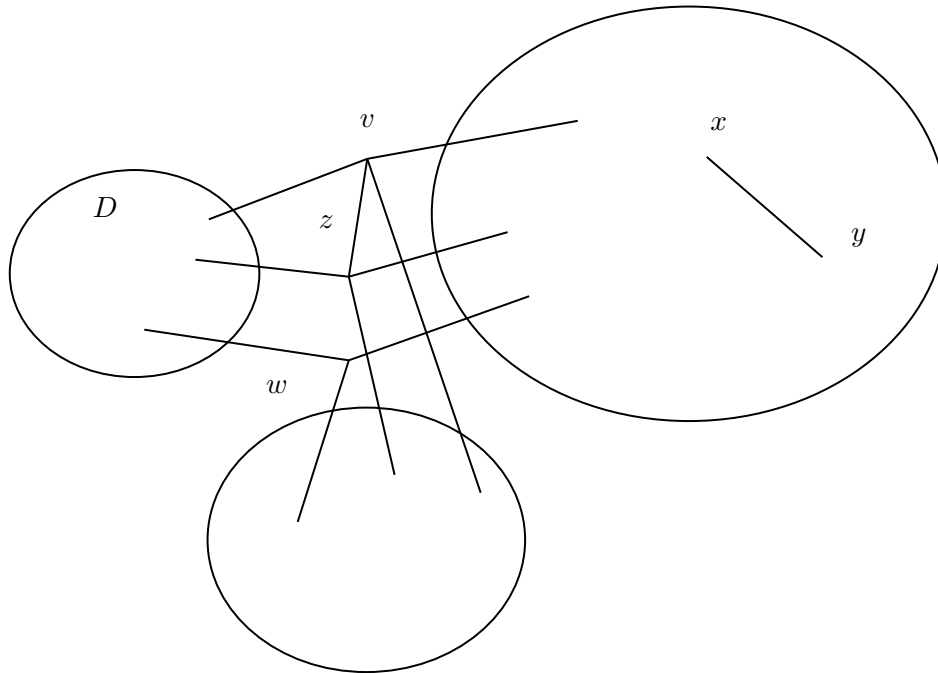
fact that G is 3-connected. Thus $\forall e(x, y) \in E, \exists v$ such that x, y, v are separators for G . Therefore x, y and v must have edges to all the connected components that get separated after removing the separators from the graph.



Let (x, y) and v be the edge-vertex combination that gives us the smallest individual connected component say C . Consider a vertex z in C which is adjacent to the vertex v . Since every edge e in G has a corresponding vertex v such that removing them makes the graph disconnected, the edge (v, z) must also have a corresponding vertex w . Consider the graph $G - \{v, z, w\}$. Since the graph gets disconnected, there must be at least two components. Since x and y are adjacent, both of them should be in one of the connected components that formed after removing $\{v, z, w\}$. There must be another component say D , which does not contain x, y .



Consider all the neighbours of z in D . All of them should belong to C and since z has been removed D must be strictly smaller to C , which contradicts our choice that x , y and v gives the smallest component.



Since we arrived at a contradiction, our initial assumption that, $\forall(x, y) \in E(G)$, the graph G will not be 3 - connected after shrinking the edge (x, y) is false. This proves that $\exists (u, v) \in E(G)$ such that the graph G remains 3-connected after shrinking the edge (u, v) . \square

Theorem 2.2 (Kuratowski's Theorem) *If a 3-connected graph G does not have $K_{3,3}$ or K_5 as a minor, then the graph is planar.*

Proof. The proof is by induction.

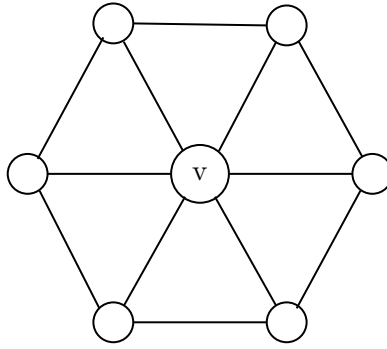
Induction on number of edges.

G is a 3-connected graph and does not have $K_{3,3}$ or K_5 as a minor.

By the previous theorem, $\exists (x, y) \in E(G)$ such that $G/(x, y)$ is 3-connected and $K_{3,3}$ and K_5 free.

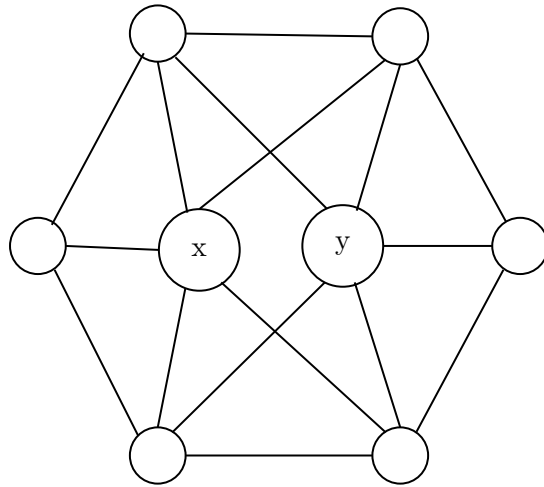
By Induction Hypothesis: $G/(x, y)$ has a planar drawing. So we need to go from $G/(x, y)$ to G and show that G is planar.

Let's consider the graph $G/(x, y)$. Let us name the vertex which we get after shrinking the edge (x, y) as v . Now let us look at the face containing the vertex v .



We can see after removing v , the graph will still be planar.

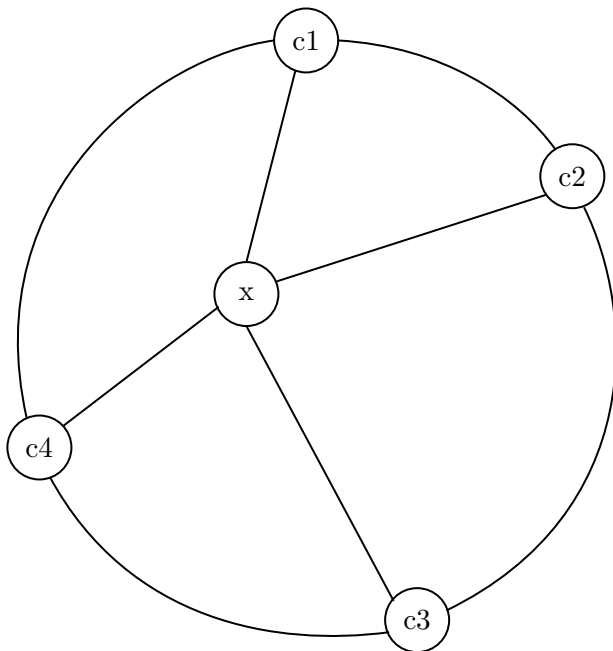
What happens if we expand v into the vertices x and y with an edge (x, y) between them?
 Let us look at a possibility below:



Looking at the above drawing, we can clearly conclude that this is not a planar drawing.

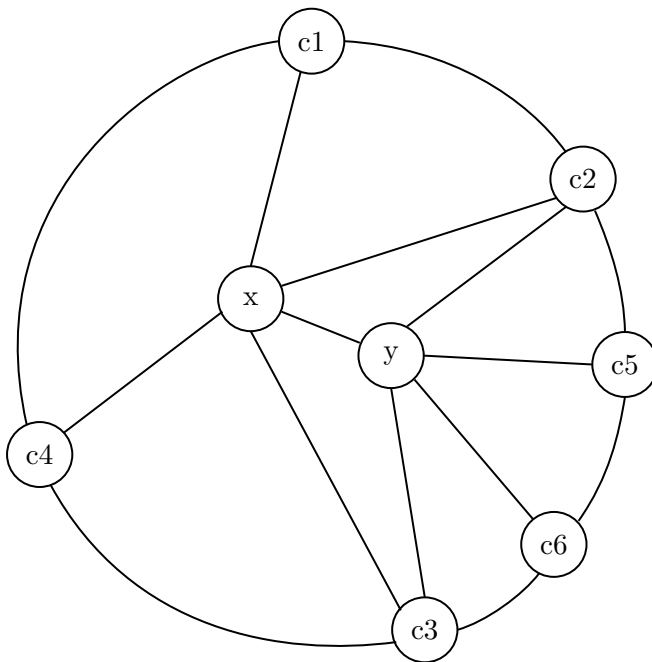
Is this really possible that expanding v into x and y will result in a non-planar drawing. We need to prove that such a case is not possible. Here's how:

Suppose x has the neighbours as $N_x = \{c_1, c_2, \dots, c_n\}$
 For example:



The set of neighbours of x forms a cycle like this.

Let N_y be the set of neighbours of y . We propose that the set N_y lies between some c_i and c_{i+1} . If this is always the case then the graph will always have a planar drawing. For example, let's say that N_y lies between c_2 and c_3 , then the graph will look like below:

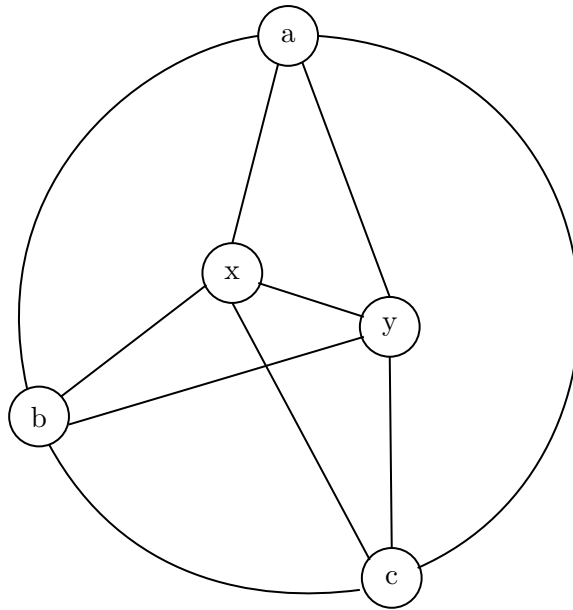


We can see that this graph is planar.

Hence we can say that if N_y lies between some c_i and c_{i+1} and $N_x \cap N_x \leq 2$, then the graph will be a planar graph.

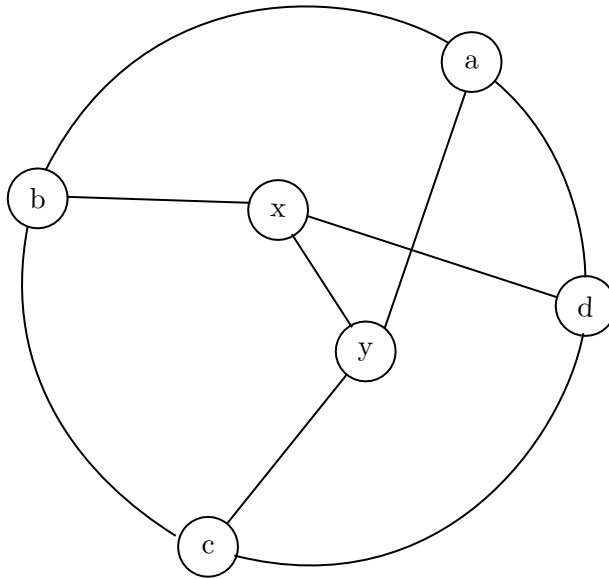
Let us see what happens if any of these two conditions is not satisfied.

Case1: If $N_x \cap N_x = 3$, we will have a graph like below:



As we can clearly see that, this graph is a K_5 , which implies that the original graph G has a K_5 as a minor, which contradicts our initial assumptions that it does not have any K_5 as a minor. Hence we can be sure that $N_x \cap N_x \leq 2$.

Case2: If $N_x \cap N_x \leq 2$, but the interval rule is not followed, i.e the neighbours of $y(N_y)$ does not lie completely between some c_i and c_{i+1} and some of the neighbours lie outside the region of vertices c_i and c_{i+1} , then the graph will have a drawing as follows:



As we can clearly see that, this graph is a $K_{3,3}$, which implies that the original graph G has a $K_{3,3}$ as a minor, which contradicts our initial assumptions that it does not have any $K_{3,3}$ as a minor. Hence we can be sure that $N_x \cap N_x \leq 2$ as well as the interval rule is followed.

This proves that the graph G will always have a planar drawing, and hence G is a planar graph.

□

2.1 References

1. *Graph Theory, Rienhard Diestel*