

## Lecture 21: COLORING

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Let's start with some definitions :

## 1 Proper k-vertex coloring of a graph

Suppose, you are given a graph  $G$  with  $n$  vertices and are asked to paint its vertices such that no two adjacent vertices have the same color. What is the minimum number of colors that you would require ? This constitutes a coloring problem. Having painted the vertices, you can group them into different sets: one set consisting of all red vertices, another of blue and so forth. This is a partitioning problem. The coloring and partitioning can be performed on edges or vertices of a graph and in case of planar graph, one may even be interested in coloring the regions.

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the **proper coloring** (or sometimes simply coloring) of a graph and a coloring using at most  $k$  colors is called a (proper)  $k$ -vertex coloring of a graph. So, a  $k$ -vertex coloring may contain fewer than  $k$  colors for  $k > 2$ .

**Definition:** A proper  $k$ -vertex coloring of a simple graph  $G = (V, E)$  is defined as a vertex coloring from a set of  $k$  colors such that no two adjacent vertices share a common color.

i.e.

$$C: V \rightarrow \{c_1, c_2, \dots, c_k\} \text{ such that} \\ \forall (u, v) \in E: C(u) \neq C(v)$$

## 2 Chromatic Number of a graph

The proper coloring which is of interest to us is one that requires the minimum number of colors. So, the smallest number of colors needed to color a graph  $G$  is called its chromatic number and it is often denoted as  $\chi(G)$ .

A graph that can be assigned a (proper)  $k$ -coloring is  $k$ -colorable, and it is  $k$ -chromatic if its chromatic number is exactly  $k$ . So,

$$\chi(G) = \min \{k \mid G \text{ has a proper } k\text{-coloring}\}$$

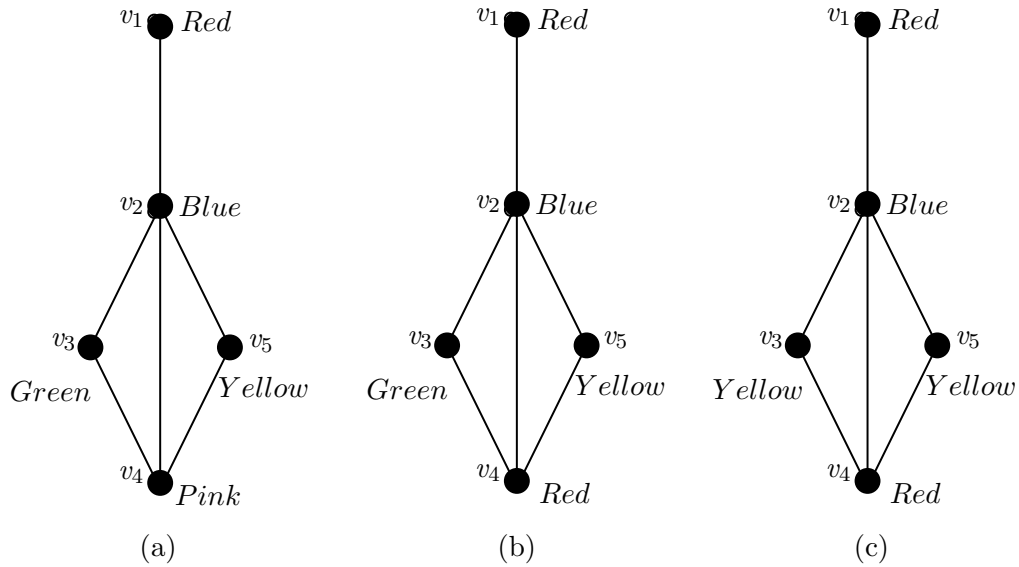


Figure 1: Proper Colorings of a graph

Figure 1 shows three different proper colorings of a graph and all three are 3– chromatic because all the three graphs need atleast 3 colors for proper coloring.

– Proper coloring of a given graph is simple enough but a proper coloring with the minimum number of colors is, in general, a difficult task. In fact, there has not yet been found a simple way of characterizing a  $k$ -chromatic graph. (The brute-force method of using all possible combinations can always be applied in any combinatorial problem but it is highly unsatisfactory because it gets out of hand as soon as the size of the graph increases beyond a few vertices.)

- **A graph  $G$  is bipartite if and only if it is 2-colorable**

*Proof.*  $\Rightarrow$ ) **If graph  $G$  is bipartite then it is 2-colorable**

Let  $G$  be bipartite graph and let the bipartition of  $G$  be  $X$  and  $Y$ . Let the vertices in  $X$  represent color 1 and the vertices in  $Y$  represent color 2. Then this satisfies the criteria for a valid coloring. Therefore,  $G$  is 2-colorable.

$\Leftarrow$ ) **If graph  $G$  is 2-colorable then it is bipartite**

Let,  $G$  be 2-colorable, i.e.,  $G$  has a 2-coloring. Now, Denote the set of all those vertices which are colored with 1 by  $V_1$  and the set of all those vertices which are colored with 2 by  $V_2$ . Then no two vertices in  $V_1$  are adjacent and no two vertices in  $V_2$  are adjacent. Thus, any edge in  $G$  joins a vertex in  $V_1$  and a vertex in  $V_2$ . Hence  $G$  is bipartite with bipartition  $V = V_1 \cup V_2$ .  $\square$

So, every 2– chromatic graph is bipartite because the coloring partitions the vertex set into two subsets  $V_1$  and  $V_2$  such that no two vertices in  $V_1$  (or  $V_2$ ) are adjacent.

Similarly, every bipartite graph is 2– chromatic, with one trivial exception; a graph of two or more isolated vertices and with no edges is bipartite but is 1– chromatic.

### 3 Independent set

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Figure 1(c) produces the partitioning

$$\{v_1, v_4\}, \{v_2\} \text{ and } \{v_3, v_5\}$$

No two vertices in any of these subsets are adjacent. Such a subset of vertices is called an independent set. More formally:

A set of vertices in a graph is said to be an *independent set* of vertices or simply an *independent set* (or an *internally stable set*) if no two vertices in the set are adjacent.

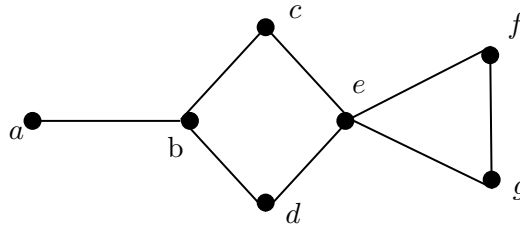


Figure-2

Here, in figure 2, the set  $\{a, c, d\}$  is an independent set. A single vertex in any graph constitutes an independent set.

A *maximal independent set* (or *maximal internally stable set*) is an independent set to which no other vertex can be added without destroying its independent property. The set  $\{a, c, d, f\}$  in figure 2, is a maximal independent set. The set  $\{b, f\}$  is another maximal independent set. The set  $\{b, g\}$  is a third one. From the preceding example, it is clear that a graph, in general, has many maximal independent sets; and they may be of different sizes. Among all maximal independent sets, one with the largest number of vertices is often of particular interest.

- The number of vertices in the largest independent set (**maximum independent set**) of a graph  $G$  is called the *independent number* (or *coefficient of internal stability*) and it is denoted by  $\alpha(G)$ . i.e.

$$\alpha(G) = \max \{ |S|, S \text{ is an independent set } \}$$

Consider a  $k$ -chromatic graph  $G$  of  $n$  vertices which is properly colored with  $k$  different colors. Since, the largest number of vertices in  $G$  with the same color cannot exceed the independent number  $\alpha(G)$ . So, we have the inequality,

$$\alpha(G) \geq \frac{n}{k}$$

i.e.

$$k.\alpha(G) \geq n$$

or,

$$\chi(G).\alpha(G) \geq n$$

Once all the maximal independent sets of  $G$  have been obtained, we find the size of the one with the largest number of vertices to get the independent number  $\alpha(G)$ . The independence number of the graph in figure 2 is four.

To find the chromatic number of  $G$ , we must find the minimum number of these maximal independent sets which collectively include all the vertices of  $G$ . For the graph in Fig. 2, sets  $\{a, c, d, f\}$ ,  $\{b, g\}$  and  $\{a, e\}$  satisfy this condition. Thus the graph is 3- chromatic.

–This method of enumerating of all maximal independent sets is inefficient and needs prohibitively large amounts of computer memory.

– The problem of finding a **maximum independent set** is called the maximum independent set problem and is an **NP-hard optimization problem**.

– Every maximum independent set also is maximal but the converse implication does not necessarily hold.

## 4 Coloring Planar Graphs

One of the main reason to the study of planar graphs back in the 1800s was the 4-color conjecture. For a given planar graph, how many colors do we need in order to color the vertices so that no two adjacent vertices have the same color (this can also be describe the coloring regions of a geographic map so that no adjacent regions get the same color).

### 4.1 6-color Theorem

**Theorem:** For a connected planar simple graph  $G$ , the vertices in  $G$  can be coloured with 6 or fewer colors. i.e., a function  $C$  exists,  $C : V(G) \rightarrow \{1, 2, 3, \dots, k\}$  where,  $1 \leq k \leq 6$  such that for every  $(u, v) \in E$ ,  $C(u) \neq C(v)$ .

i.e. Every planar graph with  $n$  vertices can be colored using at most 6 colors.

*Proof.* Let  $S(n)$  be the statement that for a connected planar simple graph  $G$  with  $n$  vertices, the vertices in  $G$  can be colored with 6 or few colors.

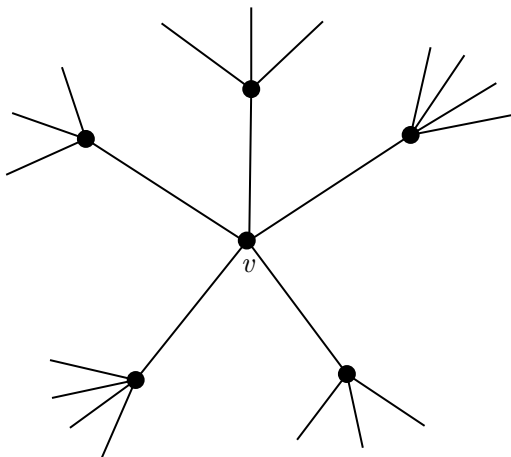
**Base Step :** For  $1 \leq n \leq 6$ , this is trivially true. A graph on 1 vertex can easily be colored with just 1 color, while a graph with 6 vertices can easily be colored with just 6

colors.

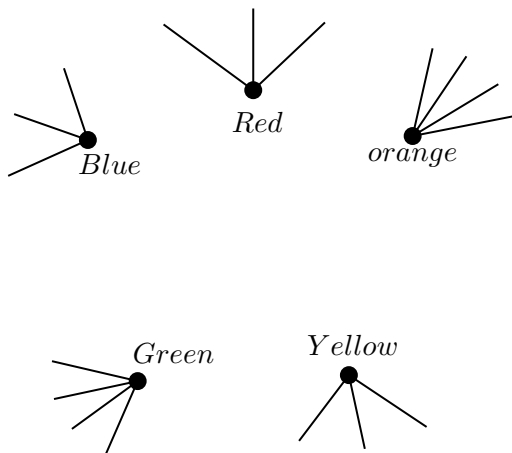
**Inductive Hypothesis :** Suppose that for all  $k \geq 2$ ,  $S(k - 1)$  is true. That is, for all connected planar simple graphs on  $k - 1$  vertices, we can obtain a good coloring of the vertices in  $G$  with 6 or fewer colors.

**Inductive Step:** We want to verify that  $S(k)$  is true (that for all connected planar simple graphs on  $k$  vertices, we can obtain a proper coloring of the vertices in  $G$  with 6 or fewer colors).

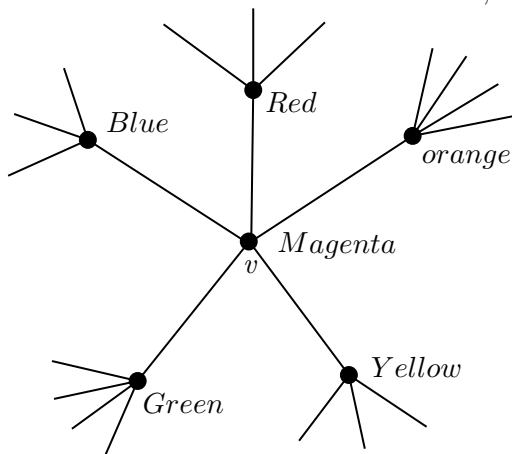
Now, let  $G$  be a connected planar simple graph on  $k$  vertices. Recall that a connected planar simple graph  $G$  contains vertices of degree 5 or less. The reason for this claim is : If  $G = (V, E)$  is a connected planar graph and  $|V| > 2$  then  $|E| \leq 3|V| - 6$  (can be proved by Euler's formula). So, average degree =  $\frac{2|E|}{|V|} \leq 6 - \frac{12}{|V|}$ . The average degree of vertices in a planar graph is strictly less than 6. So, here, graph  $G$  has a vertex  $v$  of degree at most 5. Suppose, the vertex  $v$  has  $deg(v) = 5$ .



Now, suppose that we remove vertex  $v$  and all of the edges incident with  $v$ . This graph now has less than  $k$  vertices, and by our induction hypothesis, we know this resulting graph can be colored with 6 or fewer colors.



Now, Adding vertex  $v$  back, we know that the neighbourhood of  $v$  contains 5 members. Hence if we use the 6<sup>th</sup> color for vertex  $v$ , our proof is complete.



Hence,  $S(k - 1)$  implies  $S(k)$ . By the principle of mathematical induction, for  $n \geq 1$ ,  $S(n)$  is true.  $\square$

## 4.2 5-color Theorem

**Theorem:** For a connected planar simple graph  $G$ , the vertices in  $G$  can be coloured with 5 or fewer colors. i.e., a function  $C$  exists,  $C : V(G) \rightarrow \{1, 2, 3, \dots, k\}$  where,  $1 \leq k \leq 5$  such that for every  $(u, v) \in E$ ,  $C(u) \neq C(v)$ .

i.e. Every planar graph with  $n$  vertices can be colored using at most 5 colors.

*Proof.* Let  $S(n)$  be the statement that for a connected planar simple graph  $G$  with  $n$  vertices, the vertices in  $G$  can be colored with 5 or few colors.

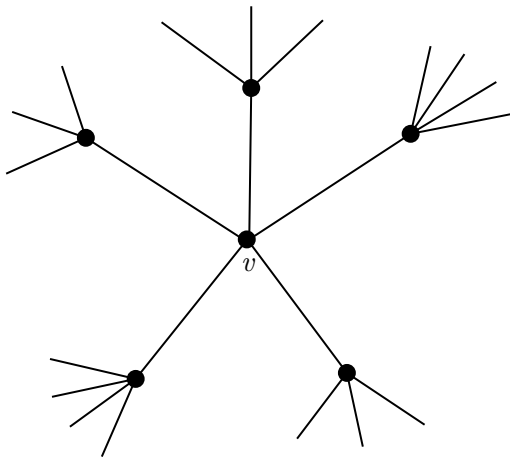
**Base Step :** For  $1 \leq n \leq 5$ , this is trivially true. A graph on 1 vertex can easily be colored with just 1 color, while a graph with 5 vertices can easily be colored with just 5 colors.

**Inductive Hypothesis :** Suppose that for all  $k \geq 2$ ,  $S(k - 1)$  is true. That is, for all connected planar simple graphs on  $k - 1$  vertices, we can obtain a coloring of the vertices

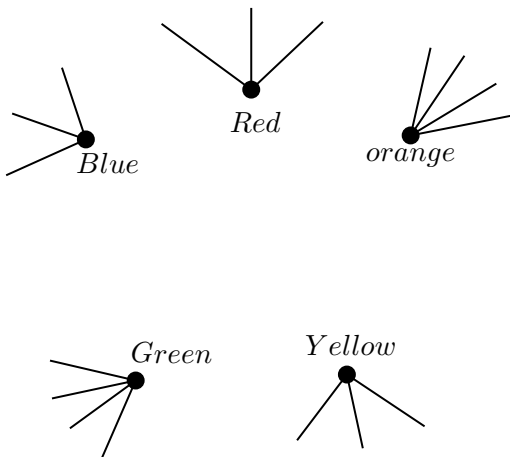
in  $G$  with 5 or fewer colors.

**Inductive Step:** We want to verify that  $S(k)$  is true (that for all connected planar simple graphs on  $k$  vertices, we can obtain a proper coloring of the vertices in  $G$  with 5 or fewer colors).

Now, once again, suppose, we have a graph  $G$  on  $k$  vertices. We know that a connected planar simple graph  $G$  contains a vertex of degree 5 or less

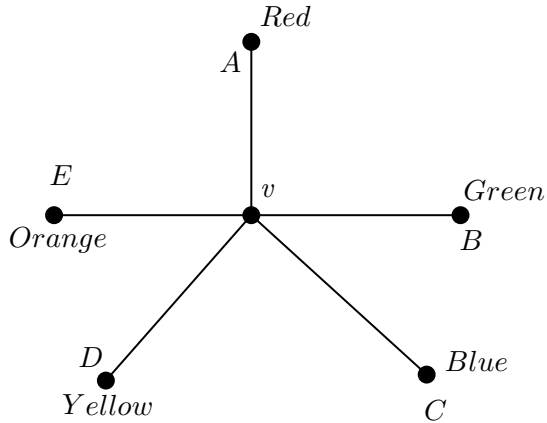


Suppose, the vertex  $v$  has  $\text{deg}(v) = 5$ . If we delete this vertex and all edges incident with  $v$  then by our induction hypothesis the resulting graph has a 5 coloring.

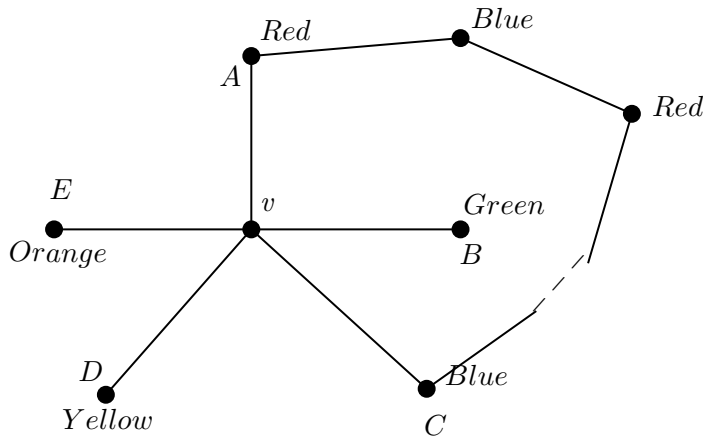


–Now, if we reinsert the vertex  $v$ . Notice that a 5-coloring cannot happen if vertex  $v$  has neighbours all with different vertex colours since  $v$  would then need a 6<sup>th</sup> color. We will prove that the neighbours of  $v$  cannot all be the same color. Now, with the following two cases. We will arbitrarily select the red and orange vertices for these cases without loss of generality.

**Case 1:** Assume there exists no path from  $A$  to  $C$ . We can change the color of  $A$  from red to blue, which is the color of  $C$ . Since no neighbor of  $v$  now has the color red, we can color  $v$  in red. We also need to change all vertices adjacent to  $A$  from blue to red. Since no path exists from  $A$  to  $C$ , the color of  $C$  remains unchanged.



**Case 2:** Assume there exists a path from  $A$  to  $C$ , alternating in color from red to blue. Note that this path bounds a planar embedding with  $B$  on the inside and  $D$  on the outside. So, there will not be any path from  $B$  to  $D$  because the given graph is planar and if we try to make the path from  $B$  to  $D$  then it will intersect the path from  $A$  to  $C$ . (This is a consequence of the *Jordan curve theorem*.) So, here, We can change the color of  $B$  from green to yellow which is the color of  $D$ . Since no neighbor of  $v$  now has the color green, we can color  $v$  in green. We also need to change all vertices adjacent to  $B$  from yellow to green. Since no path can exist from  $B$  to  $D$  (without crossing the path from  $A$  to  $C$ ), the color of  $D$  remains unchanged.



So, in both cases, we need atmost 5 colors.

Hence,  $S(k - 1)$  implies  $S(k)$ . By the principle of mathematical induction, for  $n \geq 1$ ,

$S(n)$  is true. □

### 4.3 FOUR-COLOR PROBLEM

So far we have considered proper coloring of vertices and proper coloring of edges. Now, let's briefly consider the *proper coloring of regions* in a planar graph (embedded on a plane or sphere). Just as coloring of vertices and edges, the regions of a planar graph are said to be properly colored if no two *contiguous* or *adjacent regions* have the same color. (Two regions are said to be adjacent if they have a common edge between them).

The proper coloring of regions is also called as *map coloring*, referring to the fact that in an atlas, different countries are colored such that countries with common boundaries are shown in different colors.

Once again, we are not interested in just properly coloring the regions of a given graph. We are interested in a coloring that uses the minimum number of colors.

This leads us to the most famous conjecture in graph theory. The conjecture is that every map (i.e., a planar graph) can be properly colored with four colors.

The *four-color conjecture* has been worked on by many famous mathematicians for the past 100 years.

– It has been proved that seven colors are necessary and sufficient for properly coloring maps on the surface of a torus. In fact, the Heawood map-coloring theorem gives the exact number of colors required for every orientable surface more complicated than that of a sphere.

– It has been proved that all maps containing less than 40 regions can be properly colored with four colors. Therefore, if in general the four-color conjecture is false, the counterexample has to be a very complicated and large one.

The four-color conjecture can be restated as : *Every planar graph has a chromatic number of four or less.*

– The conjecture was first proposed in 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed. One alleged proof was given by Alfred Kempe in 1879, which was widely acclaimed. another was given by Peter Guthrie Tait in 1880. It was not until 1890 that Kempe's proof was shown incorrect by Percy Heawood, and in 1891, Tait's proof was shown incorrect by Julius Petersen. Each false proof stood unchallenged for 11 years.

– During the 1960s and 1970s German mathematician Heinrich Heesch developed methods of using computers to search for a proof. Notably he was the first to use discharging (it is a technique used to prove lemmas in structural graph theory and it is most well known for its central role in the proof of the Four Color Theorem) for proving the theorem which

turned out to be important in the unavoidability portion of the subsequent Appel–Haken proof. He also expanded on the concept of reducibility and, along with Ken Durre, developed a computer test for it.

– The problem was eventually solved by Appel and Haken in 1976 when they announced that “four colors suffice”. It was the first major theorem to be proved using a computer. There is some controversy about this since their solution involves making reductions to the problem to certain cases and then using a massive computer program to check these cases. Some argue that this solution does not really count as a solution since it can’t really be checked. Others argued that this problem is a mathematics problem and should have a mathematical solution, not a computer one. Others argue that the Appel and Haken’s solution is just fine.

– If the four-color conjecture were false, there would be at least one map with the smallest possible number of regions that requires five colors. The proof showed that such a minimal counterexample cannot exist.

– Till today, no simple computer-less proof of this theorem is known!