Discrete Mathematics 30th oct, 2019

Lecture 22: Matching

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Motivation

Let us imagine some students came to our library with the name of their required books and we as librarian need to give at least one book to each students.
Let $U = the\ set\ of\ all\ students = \{S1, S2, ..., Sn\}$ and $V = The\ set\ of\ all\ books\ required\ by\ the\ students = \{B1, B2, ..., Bm\}$. So we just need to "match" one book to one student. Here we can define matching as follows:
Ler $G = (V, E)$ be a graph and $M \subseteq E$ is said to be a matching of G if $(u, v), (w, z) \in M \Rightarrow u \neq w\ and\ u \neq z\ and\ v \neq w\ and\ v \neq z$.

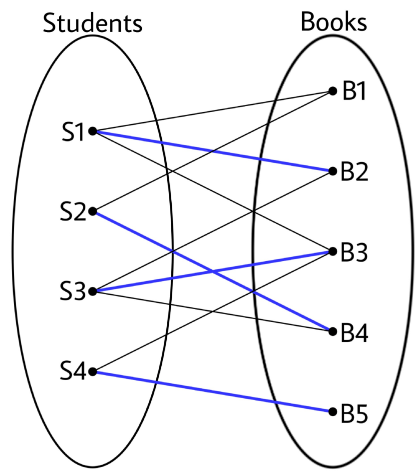

Figure 1: Graph representation where blue edges belong to a matching

Definition[1]

A matching in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are **saturated** by M; the others are **unsaturated**. A **perfect matching** in a graph is a matching that saturates every vertex.

Matching is used in commercial industries, medical purposes, transportation and so on.

In this course we are not going to study about Optimization, we just need to generate a matching as big as possible.

In a bipartite graph (U, V, E) for a matching M if $\forall u \in U \exists v \in V such\ that (u, v) \in M$ i.e. M saturate U, then M is called **U-perfect matching**. When $|U| = |V|$, U-perfect matching becomes perfect matching.

Conditions for Matching

Q. When does a bipartite graph $G = (U, V, E)$ is said to have a U-perfect matching?
Ans. Vaguely we have a necessary condition that $|V| \geq |U|$. Now, this property can be generalized as follows: For 2 vertex in U the union of neighborhoods should have atleast 2 vertices of V. If it contains 1 vertex, the matching is not possible. Similarly for any k vertices in U the neighborhood should have atleast k vertices in V. Hence, this property is necessary for every subset of U. So the necessary condition is, $\forall A \subseteq U,\ |nbd(A)| \geq |A|$. In 1935, P. Hall gave a theorem that states that it is the sufficient condition.

Hall's Marriage Theorem

If $G = (U, V, E)$ is a bipartite graph then G has a U-perfect matching if and only if $\forall A \subseteq U,\ |nbd(A)| \geq |A|$.

Proof. Necessary condition is already discussed in the above section.

Sufficiency: We will prove it by induction. We induct on the number of vertices in U.

Let us define p(n): $G = (U, V, E)$ has a U-perfect matching where $|U| = n$.

Base case: p(1) is true (trivially).

Induction Step: Let p(n) be true $\forall n \leq k - 1$. We are going to prove that p(k) is true.

Let $G = (U, V, E)$ be a bipartite graph such that $|U| = k$ and $\forall A \subseteq U,\ |nbd(A)| \geq |A|$ we are going to prove that G has a U-perfect matching.

At first let $\exists S \subset U$ such that $|nbd(S)| = |S|$.

Because $\forall A \subseteq S \Rightarrow A \subset U \Rightarrow |nbd(A)| \geq |A|$ and $|S| < k \Rightarrow$ p($|S|$) is true. Hence for p($|S|$) is true, $\exists$ a S-perfect matching M_S in induced subgraph $(S, nbd(S), E')$.

If possible let $\exists T \subset U - S$ such that $|nbd(T) - nbd(S)| < |T|$. Now consider $S \cup T \subset U$, as it is a subset of U,

$|nbd(S \cup T)| \geq |S \cup T| = |S| + |T|$, since S and T are disjoint.

Now, $|nbd(S \cup T)| = |nbd(S) \cup nbd(T)| = |nbd(S) \cup (nbd(T) - nbd(S))| = |nbd(S)| + |nbd(T) - nbd(S)| = |S| + |nbd(T) - nbd(S)|$

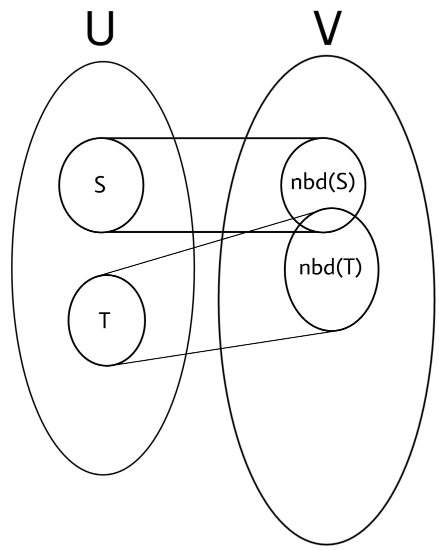

Figure 2: $G = (U, V, E)$

Combining above two expressions, We have $|S| + |nbd(T) - nbd(S)| \geq |S| + |T| \Rightarrow |nbd(T) - nbd(S)| \geq |T|$ which contradicts our assumption. Hence $\forall A \subseteq U - S,\ |nbd(A) - nbd(S)| \geq |A|$ i.e for the induced subgraph $(U - S, V - nbd(S), E'')$ the given condition is satisfied and $|U - S| < k$ hence for p($|U - S|$) is true, $\exists$ a (U-S)-perfect matching M_{U-S}. Combining M_S and M_{U-S} we will get a matching M of (U, V, E).

Now We have to show that $\exists S \subset U$ such that $|nbd(S)| = |S|$. Now if there does not exist such a set S, then we just need to delete some edges to generate a set S such that $|S| = |nbd(S)|$. □

Reference

1. Introduction to Graph Theory, Douglas B. West.