Discrete Mathematics	04/11/2019

Lecture 23: Hall's marriage and Max-Flow Min Cut Theorem

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1 First Section

Definition 1.1 (Neighborhood of a set): Let $G = (U, V, E)$ be a bipartite graph.Then the neighborhood of a set $S \subseteq U$ is denoted by $N_G(S)$ and defined by

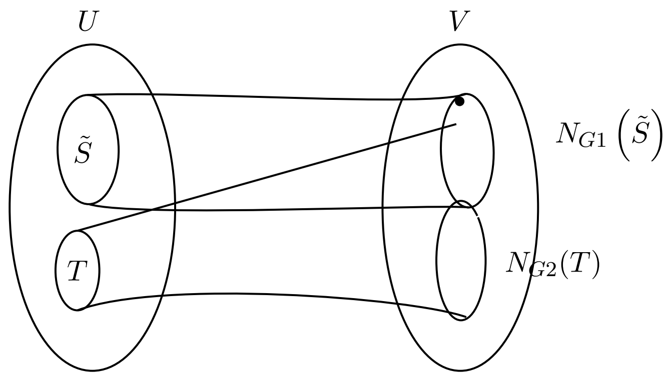
$$N_G(S) = \{v \in V : (u, v) \in E \text{ for some } u \in S\}$$

Theorem 1.1 (Hall's marriage theorem) *A bipartite graph $G = (U, V, E)$ has a U -perfect matching if and only if for every subset $S \subseteq U$, $|N_G(S)| \geq |S|$*

Proof. Suppose the graph G has a U -perfect matching. Let $S \subseteq U$. Now G has a U -perfect matching.Then for every u there is some $v \in N_G(S)$ such that $(u, v) \in E$ and if $u_1, u_2 \in U$ with $u_1 \neq u_2$ then corresponding $v_1, v_2 \in N_G(S)$ are distinct.Therefore $|N_G(S)| \geq |S|$.
Conversely assume that for every subset $S \subseteq U$, $|N_G(S)| \geq |S|$.To show G has a U -perfect matching we apply induction on m where $m = |E|$.
Base case : For $m = 1$ i.e. when there is only one edge in G, U contains only one element. So we get a U -perfect matching.
Induction hypothesis: Assume that the result holds for every $m \leq k$.
Inductive steps: We want to show the result holds for $k + 1$.
Let $G = (U, V, E)$ be a bipartite graph of $k + 1$ edges such that for every subset $S \subseteq U$, $|N_G(S)| \geq |S|$.
Case 1: For all $S \subset U$, $|N_G(S)| \geq |S| + 1$.
Let e be an edge of G. Consider $G_0 = G \backslash e$. We will show that in the graph G_0, every subset $S \subseteq U$, $|N_{G_0}(S)| \geq |S|$. Note that for $S \subseteq U$, $N_{G_0}(S) \geq N_G(S) - 1$ and $|N_G(S)| \geq |S| + 1$ therefore $N_{G_0}(S) \geq |S|$. Now G_0 contain k edges. Then by induction hypothesis, G_0 has a U -perfect matching. So G has a U -perfect matching.
Case 2: There is a $\tilde{S} \subset U$, $|N_G(\tilde{S})| = |\tilde{S}|$. Let

$$G_1 = (\tilde{S}, N_G(\tilde{S}), E|_{\tilde{S} \times N_G(\tilde{S})}) \text{ and}$$
$$G_2 = (U \setminus \tilde{S}, N_G(U \setminus \tilde{S}), E|_{U \setminus \tilde{S} \times N_G(U \setminus \tilde{S})})$$

Note that for every subset $S \subseteq \tilde{S}$, $|N_{G_1}(S)| \geq |S|$. Because $|N_G(S)| \geq |S|$ and $N_{G_1}(S) = N_G(S)$. So by induction hypothesis G_1 has a $\tilde{S}$ -perfect matching.
Now we want to show for every subset $S \subseteq U \setminus \tilde{S}$, $|N_{G_2}(S)| \geq |S|$. If possible let there exist $T \subseteq U \setminus \tilde{S}$, $|N_{G_2}(T)| < |T|$. Now consider $\tilde{S} \cup T$ in G. Then

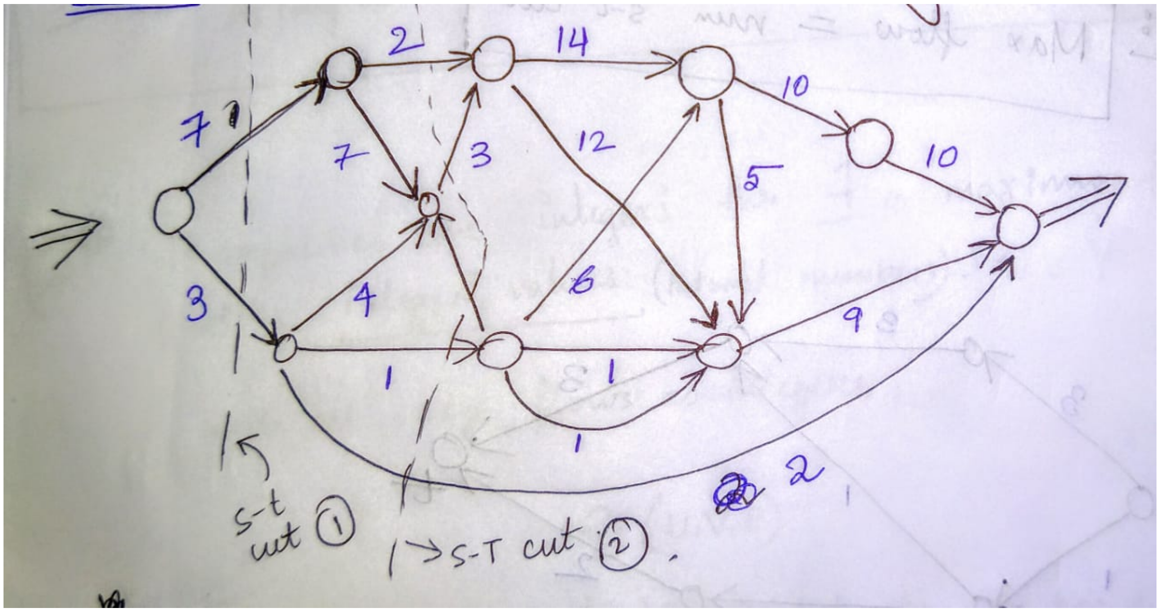


$|N_G(\tilde{S} \cup T)| = |N_{G_1}(\tilde{S}) \cup N_{G_2}(T)| < |\tilde{S}| + |T| = |\tilde{S} \cup T|$ as $\tilde{S} \cap T = \phi$, which is a contradiction.
So for every subset $S \subseteq U \setminus \tilde{S}$, $|N_{G_2}(S)| \geq |S|$. By induction hypothesis G_2 has a $U \setminus \tilde{S}$ -perfect matching. Thus by considering the graphs G_1, G_2 we can conclude that G has U -perfect matching. This completes the proof. $\square$

2 Second Section

Flow

We are given a directed Graph and a start node **'s'** and sink node **'t'**. and in the graph each directed edge has a weight associated with it, which determines how much flow can go through it(zero flow is represented by no edge). the node **'s'** has a inflow capacity of infinity and the node **'t'** has an outflow capacity of infinity. Our goal is to find given this network what is the maximum flow that can be achieved from **'s'** to **'t'**



$F : E \rightarrow R^{+}$ $\forall v \in$ V-s,t:

Kirchoff's law is satisfied for all v in V other than **s** and **t**:

$$\sum_u F(u,v) = \sum_w F(v,w)$$

we want to maximise the flow i.e. Σ_uF(s,u)

2.1 $s - t$ cut

S: subset of vertices in V such that the vertices **s** is in the subset
T: subset of vertices in V such that the vertices **t** is in the subset

V=$S \bigcup T$

$\Sigma weight(u,v)$ such that $u \in$S *and* v$\in$T.

This is the capacity of the cut. i.e sum of all the weights of all the edges going across the cut from S $\rightarrow T$

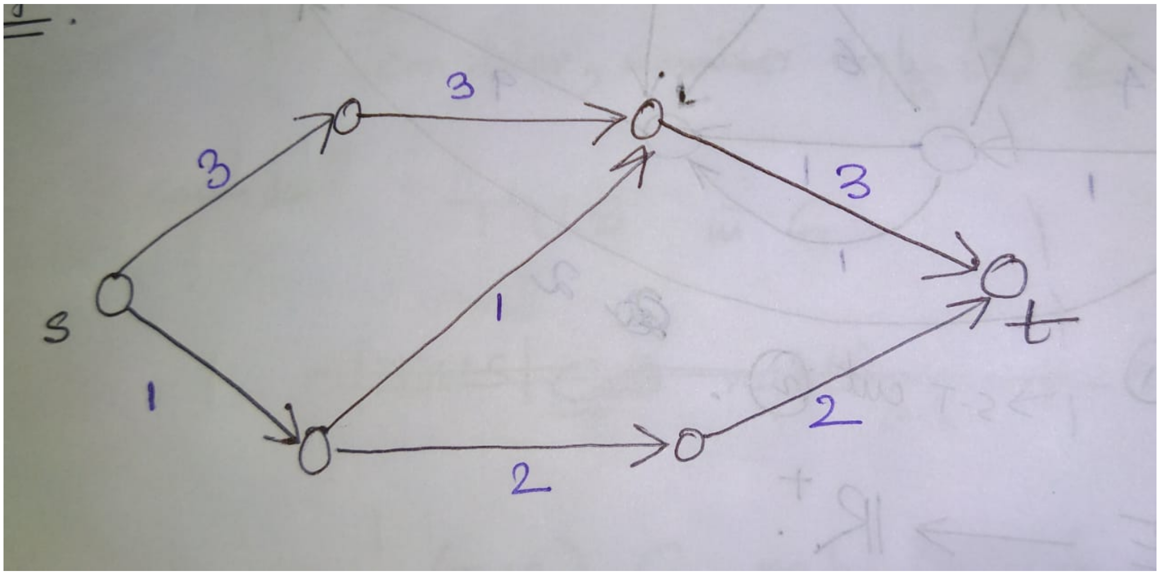
Now, it can easily be seen that

Max Flow$\leq$ $capacity of any$ s$-$tcut.

$\Rightarrow MaxFLow \leq min(s-$t $cut)$.

2.2 Algorithm to find Max Flow

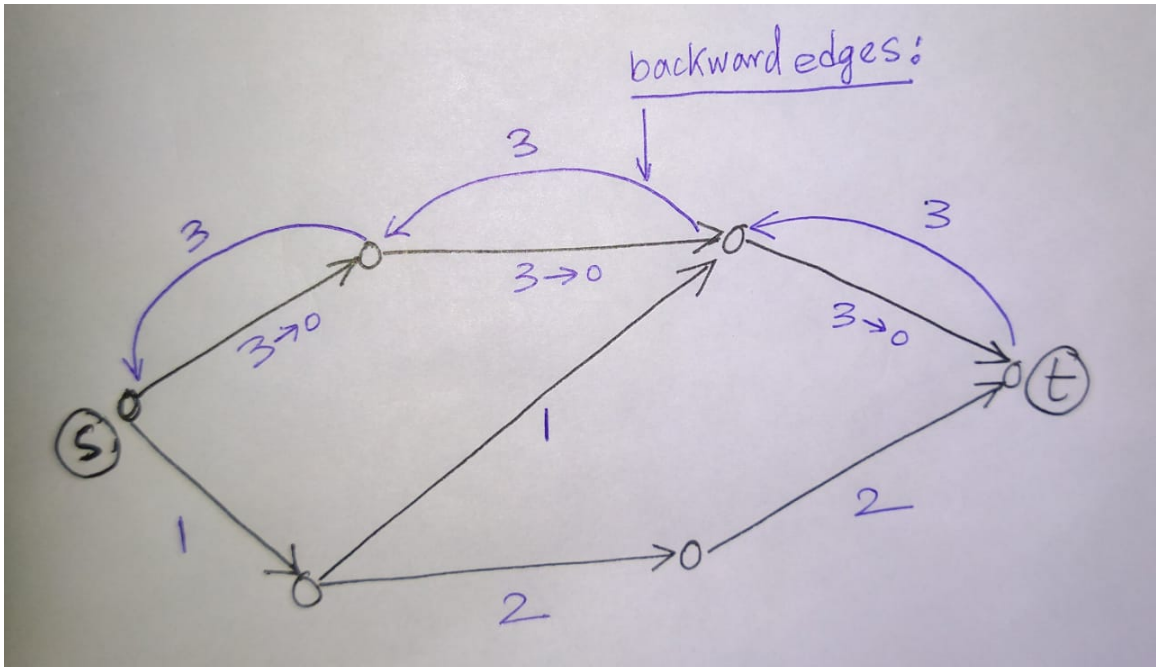
Given graph:



Step-1 :

Every time we find a path from s to t we send as much flow as possible through the path and add a backwards edge for each of the edges in the path used having same weight as the max flow sent through that path. And as for the forward edges we subtract the flow from the edge capacity and assign that as the new weight of that edge

Let us now send a flow of '3' through the path 3-3-3 as seen in the above graph and add backward edges as per our step-1. we get as shown below:



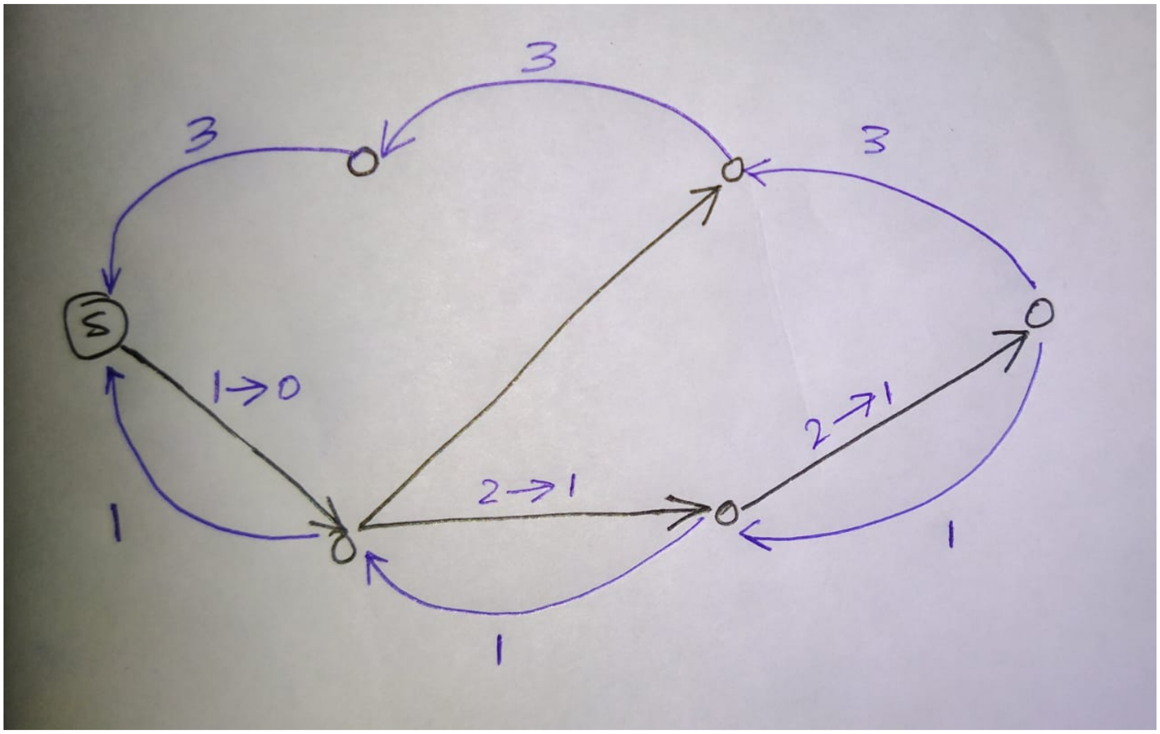
We can remove the forward edges for the previous path as the new edge capacity is zero (which means no edge)

and we have the new Graph as :

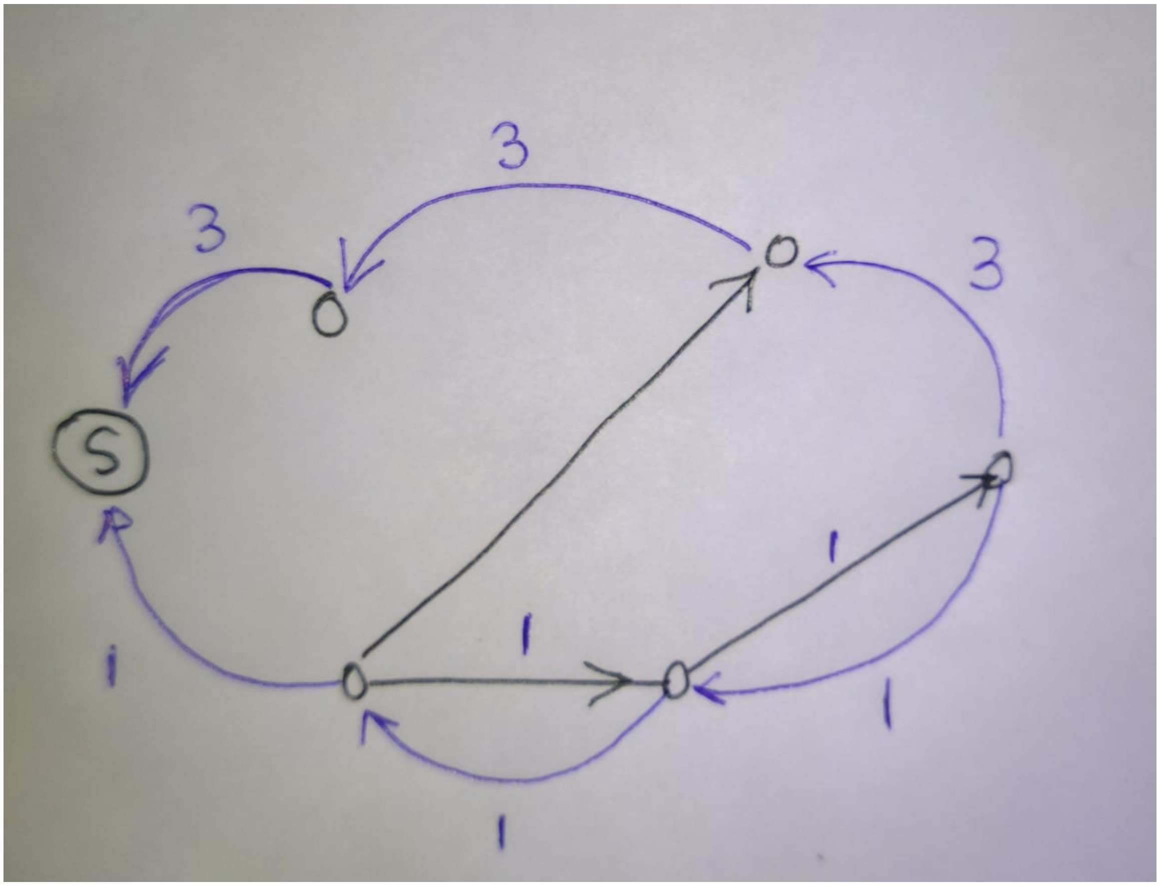
Step-2: We repeatedly do the above step-1 till we get no more paths from s to t.

(in our choosen example we get the following):

sending flow of '1' thorugh the path 1-2-2 and we get:



We stop here as no more path is there from s to t,



Therefore we were able to send '3+1=4' flow from s to t. hence the max flow for our Graph is 4.